An inverse problem for Moore Gibson Thompson equation

arising in high intensity ultrasound

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Joint work with

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- **Sebastian Zamorano**, Universidad de Santiago.

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- Henceforth we will consider $\tau = 1$.

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- If $b = 0$, there does not exist an infinitesimal generator of the semigroup.
- If $\gamma := \alpha - \frac{c^2}{b} > 0$, the group associated to the equation is exponentially stable, and for $\gamma = 0$, the group is conservative.

Inverse Problem

The inverse problem is to recover the unknown coefficient $\alpha(x)$

$$\alpha(x) \to \begin{cases} u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b\Delta u_t = f \\ u = g \\ u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2 \end{cases} \to u(\alpha)$$

from partial knowledge of some trace of the solution $u(\alpha)$ at the boundary, where $\Gamma_0 \subset \partial\Omega$ is a relatively open subset, called the observationregion, and n is the outward unit normal vector on Γ.

$$\alpha \to \frac{\partial u(\alpha)}{\partial n} \text{ on } \Gamma_0 \times (0, T),$$

The aim tasks

Under appropriate hypotheses:

- **Uniqueness:**

$$\frac{\partial u(\alpha_1)}{\partial n} = \frac{\partial u(\alpha_2)}{\partial n} \text{ on } \Gamma_0 \times (0, T) \text{ implies } \alpha_1 = \alpha_2 \text{ in } \Omega.$$

- **Stability:**

$$\|\alpha_1 - \alpha_2\|_{X(\Omega)} \leq C \left\| \frac{\partial u(\alpha_1)}{\partial n} - \frac{\partial u(\alpha_2)}{\partial n} \right\|_{Y(\Gamma_0)},$$

  for some appropriate spaces $X(\Omega)$ and $Y(\Gamma_0)$.

- **Reconstruction:** Design an algorithm to recover the coefficient α from the knowledge of $\frac{\partial u(\alpha)}{\partial n}$ on Γ_0.

Difficulties

- Third-order in time.
- Energies is not preserved, α represent a dissipation coefficient.
- M-G-T is not controllable with interior control.
- Improve the energies estimates.

Under certain conditions for α, Γ_0 and the time T.

The admissible coefficients:

$$\mathcal{A}_M = \left\{ \alpha \in L^\infty(\Omega), \quad \frac{c^2}{b} \leq \alpha(x) \leq M \quad \forall x \in \overline{\Omega} \right\}, \tag{4}$$

and we consider the assumptions:

$$\exists x_0 \notin \Omega \text{ such that } \Gamma_0 \supset \{x \in \Gamma : \ (x - x_0) \cdot n \geq 0\}, \tag{5}$$

and

$$T > \sup_{x \in \Omega} |x - x_0|. \tag{6}$$

Also, we suppose that the data satisfies

$$\begin{gathered} (u_0, u_1, u_2) \in (L^2(\Omega) \times H^{-1}(\Omega) \times H^{-2}(\Omega)), \quad |u_2| \geq \eta > 0, \\ f \in L^1(0, T; L^2(\Omega), \quad g \in L^2(0, T; L^2(\partial\Omega)). \end{gathered} \tag{7}$$

Principal Results

Our main result, concerning the stability, is the following:

Theorem (R. L., A. Mercado, S. Zamorano)

Suppose that $\Gamma_0 \subset \partial\Omega$ *and* $T > 0$ *satisfy* (5)-(6) *and the data satisfy* (7)*. Let* $M > 0$*, and* $\alpha_2 \in \mathcal{A}_M$ *be such that the corresponding solution* $u(\alpha_2)$ *of* (3) *(with* $\alpha = \alpha_2$*) satisfies*

$$u(\alpha_2) \in H^3(0, T; L^\infty(\Omega)).$$

Then there exists a constant $C > 0$ *such that*

$$C^{-1}\|\alpha_1 - \alpha_2\|^2_{L^2(\Omega)} \leq \left\| \frac{\partial u(\alpha_1)}{\partial n} - \frac{\partial u(\alpha_2)}{\partial n} \right\|^2_{H^2(0,T;L^2(\Gamma_0))} \leq C\|\alpha_1 - \alpha_2\|^2_{L^2(\Omega)} \tag{8}$$

for all $\alpha_1 \in \mathcal{A}_M$*.*

Remarks

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- The hypotheses (5) and (6) on Γ_0 and T typically arises in the study of stability or observability inequalities for the wave equation.
- The assumption of the positiveness for u_2 appearing in Theorem 1 is classical when applying the Bukhgeim-Klibanov method and Carleman estimates for inverse problems with only one boundary measurement.

Well–posedness

Theorem ([1], Theorem 2.2)

Let $b > 0$ *and* $\alpha \in L^\infty(\Omega)$. *Then the solution* $u(\alpha)$ *is generated by a strongly continuous group on the state space*

$$H = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega).$$

That is, for each $(u_0, u_1, u_2) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$ *and* $f \in L^1(0, T; L^2(\Omega))$, *there exists a unique solution* $U = (u(\alpha), u_t(\alpha), u_{tt}(\alpha)) \in C([0, T]; H)$.

Kaltenbacher, Barbara and Lasiecka, Irena

Exponential decay for low and higher energies in the third order linear Moore-Gibson-Thompson equation with variable viscosity

Palest. J. Math 1 (2012) 1–10.

Hidden regularity

Theorem (R. L., A. Mercado, S. Zamorano)

The unique solution
$(u, u_t, u_{tt}) \in C([0,T]; (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega))$ *of* (3) *satisfies*

$$\frac{\partial u}{\partial n} \in H^1(0,T; L^2(\partial\Omega)). \tag{9}$$

Moreover, the normal derivative satisfies

$$\left\| \frac{\partial u}{\partial n} \right\|^2_{H^1(0,T;L^2(\partial\Omega))} \leq C(\|u_0\|^2_{H^2(\Omega)\cap H_0^1(\Omega)} + \|u_1\|^2_{H_0^1(\Omega)} + \|u_2\|^2_{L^2(\Omega)} + \|f\|^2_{L^1(0,T;L^2(\Omega))}). \tag{10}$$

Consequently, the mapping $(f, u_0, u_1, u_2) \mapsto \frac{\partial u}{\partial n}$ *is linear continuous from* $L^1(0,T; L^2(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega))$ *into* $H^1(0,T; L^2(\partial\Omega))$.

Carleman estimates

For $x_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ and $\lambda > 0$, we define the weight functions ϕ and φ_λ as follows

$$\varphi_\lambda(x,t) = e^{\lambda \phi(x,t)}, \tag{11}$$

where

$$\phi(x,t) = |x - x_0|^2 - \beta t^2 + M_0, \quad 0 < \beta < 1, \tag{12}$$

and M_0 is chosen such that

$$\forall (x,t) \in \Omega \times (-T,T), \quad \phi(x,t) \geq 1. \tag{13}$$

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Suppose that Γ_0 *and* T *satisfies* (5)–(6). *Let* $M > 0$ *and* $\alpha \in \mathcal{A}_M$. *Let* $\beta \in (0,1)$ *such that*

$$\beta T > \sup_{x \in \Omega} \|x - x_0\|. \tag{11}$$

Then, there exists $s_0 > 0$, $\lambda > 0$ *and a positive constant* C *such that for all* $s \geq s_0$

$$\begin{aligned}
&\sqrt{s} \int_\Omega e^{2s\varphi_\lambda(0)} |y_{tt}(0)|^2 dx \\
&+ s\lambda c^4 \int_0^T \int_\Omega e^{2s\varphi_\lambda} \varphi_\lambda (|y_t|^2 + |\nabla y|^2) dxdt + s^3\lambda^3 c^4 \int_0^T \int_\Omega e^{2s\varphi_\lambda} \varphi_\lambda^3 |y|^2 dxdt \\
&+ s\lambda \int_0^T \int_\Omega e^{2s\varphi_\lambda} \varphi_\lambda (|y_{tt}|^2 + |\nabla y_t|^2) dxdt + s^3\lambda^3 \int_0^T \int_\Omega e^{2s\varphi_\lambda} \varphi_\lambda^3 |y_t|^2 dxdt \\
&\leq C \int_0^T \int_\Omega e^{2s\varphi_\lambda} |f|^2 dxdt + Cs\lambda \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left(|\nabla y_t \cdot n|^2 + c^4 |\nabla y \cdot n|^2 \right) d\sigma dt,
\end{aligned}$$

for all $y \in L^2(0, T; H_0^1(\Omega))$ *satisfying* $f \in L^2(\Omega \times (0, T))$, $y(\cdot, 0) = y_t(\cdot, 0) = 0$ *in* Ω, *and* $y_{tt}(\cdot, 0) \in L^2(\Omega)$.

Sketch of the proof for Carleman estimates

For the wave equation we have

$$E_w(u) := \sqrt{s}\int_\Omega e^{2s\varphi_\lambda(0)}|u_t(0)|^2dx + s\lambda\int_0^T\int_\Omega e^{2s\varphi_\lambda}\varphi_\lambda(|u_t|^2+|\nabla u|^2)dxdt + s^3\lambda^3\int_0^T\int_\Omega e^{2s\varphi_\lambda}\varphi_\lambda^3|u|^2dxdt$$
$$\leq C\int_0^T\int_\Omega e^{2s\varphi_\lambda}|L_0u|^2dxdt + Cs\lambda\int_0^T\int_{\Gamma_0} e^{2s\varphi_\lambda}\left(|\nabla u\cdot n|^2\right)d\sigma dt,$$

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where $L_0u = u_{tt} - b\Delta u$ is the classical wave operator. We consider the operator $L_\alpha u = u_{ttt} + \alpha u_{tt} - c^2\Delta u - b\Delta u_t$, and we have

$$L_\alpha u = L_0u_t + \frac{c^2}{b}L_0u + (\alpha - \frac{c^2}{b})u_{tt}.$$

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Now we consider the weight norm $\|f\|_w^2 = \int_0^T\int_\Omega e^{2s\varphi_\lambda}|f|^2dxdt$. And we compute

$$\|L_\alpha u - (\alpha - \frac{c^2}{b})u_{tt}\|_w^2 = \|L_0u_t\|_w^2 + \frac{c^4}{b^2}\|L_0u\|_w^2 + 2\frac{c^2}{b}(e^{2s\varphi_\lambda}\partial_t L_0u, L_0u).$$

Using the Carleman estimates for the waver equation,

$$C\|L_\alpha u\|_w^2 + C\|(\alpha - \frac{c^2}{b})u_{tt}\|_w^2 \geq E_w(u_t) + \frac{c^4}{b^2}E_w(u) + \frac{c^2}{b}\int_0^T\int_\Omega e^{2s\varphi_\lambda}\partial_t|L_0u|^2dxd$$

Sketch of the proof for the stability Theorem

Bukhgeim-Klibanov method.

Algorithm to find the coefficient α

This algorithm will be based by the work of Baudouin, Buhan and Ervedoza [**?**], in which they propose a reconstruction algorithm for the potential of the wave equation.
Let us consider the following functional

$$
\begin{aligned}
J[\mu, f](y) =& \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi_\lambda} |L_\alpha y - f|^2 dxdt \\
&+ \frac{1}{2}\int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left(\left| \frac{\partial y}{\partial n} - \mu \right|^2 + \left| \frac{\partial y_t}{\partial n} - \mu_t \right|^2 \right) d\sigma dt, \quad (12)
\end{aligned}
$$

where $\alpha \in \mathcal{A}_M$, $g \in L^2(\Omega \times (0, T))$, $\mu \in H^1(0, T; L^2(\Gamma_0))$.

L. Baudouin, M. De Buhan, and S. Ervedoza.
Global Carleman estimates for waves and applications.
Communications in Partial Differential Equations, 38(5):823–859, 2013.

Algorithm:

1. **Initialization:** $\alpha^0 = \frac{c^2}{b}$.
2. **Iteration:** From k to $k+1$

Step 1 - Given α^k we consider $\mu^k = \partial_t \left(\frac{\partial u(\alpha^k)}{\partial n} - \frac{\partial u(\alpha)}{\partial n} \right)$ on $\Gamma_0 \times (0,T)$

where $u(\alpha^k)$ and $u(\alpha)$ are the solution of the problems

$$\begin{cases} L_{\alpha^k} u = f, \Omega \times (0,T) \\ u = g, \partial\Omega \times (0,T) \\ u(\cdot,0) = u_0, u_t(\cdot,0) = u_1, u_{tt}(\cdot,0) = u_2, \Omega \end{cases} \tag{13}$$

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Step 2 - Minimize the functional $J[\mu^k, 0]$ on the admissible trajectories y.

Step 3 - Let $y^{*,k}$ the minimizer of $J[\mu^k, 0]$ and

$$\widetilde{\alpha}^{k+1} = \alpha^k + \frac{y_{tt}^{*,k}(\cdot,0)}{u_2}. \tag{15}$$

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Step 4 - Finally, consider $\alpha^{k+1} = T(\widetilde{\alpha}^{k+1})$, where

$$T(\alpha) = \begin{cases} M & \text{if } \alpha > M \\ \alpha & \text{if } \frac{c^2}{b} \leq \alpha \leq M \\ \frac{c^2}{b} & \text{if } \alpha < \frac{c^2}{b}. \end{cases} \tag{14}$$

This function T is to guarantee at each step that α^k belongs to the admissible set $\mathcal{A}_M$.

The convergence of this algorithm

Theorem (R. L., A. Mercado, S. Zamorano)

Assume the same hypotheses of observability Theorem, and the following assumption of $u(\alpha)$:

$$u(\alpha) \in H^3(0, T; L^\infty(\Omega)) \text{ and } |u_2| \geq \eta > 0. \tag{15}$$

Then, there exists a constant $C > 0$ *and* $s_0 > 0$ *such that for all* $s \geq s_0$ *and* $k \in \mathbb{N}$

$$\int_\Omega e^{2s\varphi_\lambda(0)}(\alpha^{k+1} - \alpha)^2 dx \leq \frac{C}{\sqrt{s}} \int_\Omega e^{2s\varphi_\lambda(0)}(\alpha^k - \alpha)^2 dx. \tag{16}$$

Working in progress

Consider a new numerical approach

Baudouin, L. and de Buhan, M. and Ervedoza, S.

Convergent Algorithm Based on Carleman Estimates for the Recovery of a Potential in the Wave Equation.

SIAM Journal on Numerical Analysis, 55(4):1578-1613, 2017.

Thank you for your attention!