An inverse problem for Moore Gibson Thompson equation arising in high intensity ultrasound

Rodrigo Lecaros

Universidad Técnica Federico Santa María

Workshop on Applied & Interdisciplinary Mathematics
19-20-21 March, 2019
supported by FONDECYT project 11180874
Joint work with

- **Alberto Mercado**, Universidad Técnica Federico Santa María.
- **Sebastian Zamorano**, Universidad de Santiago.
The model

- A model for wave propagation in viscous thermally relaxing fluids.
The model

- A model for wave propagation in viscous thermally relaxing fluids.
- It is well known that the use the classical Fourier’s law to describe the heat flux leads to an infinite signal speed paradox.
A model for wave propagation in viscous thermally relaxing fluids.

It is well known that the use the classical Fourier’s law to describe the heat flux leads to an infinite signal speed paradox.

Moore Gibson Thompson (MGT) equation

\[
\begin{cases}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = f, & \Omega \times (0, T) \\
u = g, & \partial \Omega \times (0, T) \\
u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{cases}
\]

In this work, we consider the case \( \alpha = \alpha(x) \) and \( b > 0 \).
The model

Moore Gibson Thompson (MGT) equation

\[
\begin{aligned}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t &= f, & \Omega \times (0, T) \\
\xi &= g, & \partial \Omega \times (0, T) \\
u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{aligned}
\]

(2)

In this work, we consider the case \( \alpha = \alpha(x) \) and \( b > 0 \).

- \( \alpha(x) > 0 \), is a coefficient depending on a viscosity of the fluid.
The model

Moore Gibson Thompson (MGT) equation

\[
\begin{aligned}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t &= f, & \Omega \times (0, T) \\
u &= g, & \partial \Omega \times (0, T) \\
u(\cdot, 0) &= u_0, \ u_t(\cdot, 0) = u_1, \ u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{aligned}
\]

In this work, we consider the case \( \alpha = \alpha(x) \) and \( b > 0 \).

- \( \alpha(x) > 0 \), is a coefficient depending on a viscosity of the fluid.
- \( \tau \) is the relaxation time.
The model

Moore Gibson Thompson (MGT) equation

\[
\begin{cases}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = f, & \Omega \times (0, T) \\
u = g, & \partial \Omega \times (0, T) \\
u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{cases}
\]

In this work, we consider the case \(\alpha = \alpha(x)\) and \(b > 0\).

- \(\alpha(x) > 0\), is a coefficient depending on a viscosity of the fluid.
- \(\tau\) is the relaxation time.
- \(c\) is the speed of sound.
The model

Moore Gibson Thompson (MGT) equation

\[
\begin{align*}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t &= f, & \Omega \times (0, T) \\
u &= g, & \partial \Omega \times (0, T) \\
u(\cdot, 0) &= u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{align*}
\]

(2)

In this work, we consider the case \( \alpha = \alpha(x) \) and \( b > 0 \).

- \( \alpha(x) > 0 \), is a coefficient depending on a viscosity of the fluid.
- \( \tau \) is the relaxation time.
- \( c \) is the speed of sound
- \( b = \delta + \tau c^2 \), where \( \delta \geq 0 \) is the diffusivity of sound.
The model

Moore Gibson Thompson (MGT) equation

\[
\begin{aligned}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t &= f, & \Omega \times (0, T) \\
u &= g, & \partial \Omega \times (0, T) \\
u(\cdot, 0) &= u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{aligned}
\]

(2)

In this work, we consider the case \( \alpha = \alpha(x) \) and \( b > 0 \).

- \( \alpha(x) > 0 \), is a coefficient depending on a viscosity of the fluid.
- \( \tau \) is the relaxation time.
- \( c \) is the speed of sound
- \( b = \delta + \tau c^2 \), where \( \delta \geq 0 \) is the diffusivity of sound.
- Henceforth we will consider \( \tau = 1 \).
The model

Moore Gibson Thompson (MGT) equation

\[
\begin{aligned}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t &= f, & \Omega \times (0, T) \\
u &= g, & \partial \Omega \times (0, T) \\
u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{aligned}
\]

The equation models different phenomena depending on the parameters.
The model

### Moore Gibson Thompson (MGT) equation

\[
\begin{cases}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = f, & \Omega \times (0, T) \\
u = g, & \partial \Omega \times (0, T) \\
u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{cases}
\]

1. The equation models different phenomena depending on the parameters.
2. If \( b = 0 \) and \( f = \beta (u^2)_t \) is the Westervelt equation, which is used as a model of finite-amplitude nonlinear wave propagation in soft tissues.
The model

Moore Gibson Thompson (MGT) equation

\[
\begin{aligned}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t &= f, & \Omega \times (0, T) \\
\quad u &= g, & \partial \Omega \times (0, T) \\
\quad u(\cdot, 0) &= u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{aligned}
\]

- The equation models different phenomena depending on the parameters.
- If \(b = 0\) and \(f = \beta (u^2)_t\) is the Westervelt equation, which is used as a model of finite-amplitude nonlinear wave propagation in soft tissues.
The equation models different phenomena depending on the parameters. 

- If $b = 0$ and $f = \beta (u^2)_t$ is the Westervelt equation, which is used as a model of finite-amplitude nonlinear wave propagation in soft tissues.
- If $b > 0$, the well–posedness and exponential decay of the equation has been proved by Kaltenbacher et al.
The model

**Moore Gibson Thompson (MGT) equation**

\[
\begin{cases}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = f, & \Omega \times (0, T) \\
u = g, & \partial \Omega \times (0, T) \\
u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{cases}
\]

- The equation models different phenomena depending on the parameters.
- If \( b = 0 \) and \( f = \beta(u^2)_t \) is the Westervelt equation, which is used as a model of finite-amplitude nonlinear wave propagation in soft tissues.
- If \( b > 0 \), the well–posedness and exponential decay of the equation has been proved by Kaltenbacher et al.
- If \( b = 0 \), there does not exist an infinitesimal generator of the semigroup.
The model

Moore Gibson Thompson (MGT) equation

\[
\begin{aligned}
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t &= f, & \Omega \times (0, T) \\
u &= g, & \partial \Omega \times (0, T) \\
u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, & \Omega,
\end{aligned}
\]

- The equation models different phenomena depending on the parameters.
- If \( b = 0 \) and \( f = \beta (u^2)_t \) is the Westervelt equation, which is used as a model of finite-amplitude nonlinear wave propagation in soft tissues.
- If \( b > 0 \), the well–posedness and exponential decay of the equation has been proved by Kaltenbacher et al.
- If \( b = 0 \), there does not exist an infinitesimal generator of the semigroup.
- If \( \gamma := \alpha - \frac{c^2}{b} > 0 \), the group associated to the equation is exponentially stable, and for \( \gamma = 0 \), the group is conservative.
The inverse problem is to recover the unknown coefficient $\alpha(x)$

$$\alpha(x) \rightarrow \begin{cases} 
    u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = f \\
    u = g \\
    u(\cdot, 0) = u_0, \ u_t(\cdot, 0) = u_1, \ u_{tt}(\cdot, 0) = u_2 
\end{cases} \rightarrow u(\alpha)$$

from partial knowledge of some trace of the solution $u(\alpha)$ at the boundary, where $\Gamma_0 \subset \partial \Omega$ is a relatively open subset, called the observation region, and $n$ is the outward unit normal vector on $\Gamma$.

$$\alpha \rightarrow \frac{\partial u(\alpha)}{\partial n} \text{ on } \Gamma_0 \times (0, T),$$
The aim tasks

Under appropriate hypotheses:

- **Uniqueness:**

  \[
  \frac{\partial u(\alpha_1)}{\partial n} = \frac{\partial u(\alpha_2)}{\partial n}
  \]
  on \( \Gamma_0 \times (0, T) \) implies \( \alpha_1 = \alpha_2 \) in \( \Omega \).

- **Stability:**

  \[
  \| \alpha_1 - \alpha_2 \|_{X(\Omega)} \leq C \left\| \frac{\partial u(\alpha_1)}{\partial n} - \frac{\partial u(\alpha_2)}{\partial n} \right\|_{Y(\Gamma_0)},
  \]
  for some appropriate spaces \( X(\Omega) \) and \( Y(\Gamma_0) \).

- **Reconstruction:** Design an algorithm to recover the coefficient \( \alpha \) from the knowledge of \( \frac{\partial u(\alpha)}{\partial n} \) on \( \Gamma_0 \).
Difficulties

- Third-order in time.
- Energies is not preserved, $\alpha$ represent a dissipation coefficient.
- M-G-T is not controllable with interior control.
- Improve the energies estimates.
Under certain conditions for $\alpha$, $\Gamma_0$ and the time $T$.

The admissible coefficients:

$$A_M = \left\{ \alpha \in L^\infty(\Omega), \quad \frac{c^2}{b} \leq \alpha(x) \leq M \quad \forall x \in \overline{\Omega} \right\},$$

(4)

and we consider the assumptions:

$$\exists x_0 \notin \Omega \text{ such that } \Gamma_0 \supset \{ x \in \Gamma : (x - x_0) \cdot n \geq 0 \},$$

(5)

and

$$T > \sup_{x \in \Omega} |x - x_0|.$$

(6)

Also, we suppose that the data satisfies

$$(u_0, u_1, u_2) \in (L^2(\Omega) \times H^{-1}(\Omega) \times H^{-2}(\Omega)), \quad |u_2| \geq \eta > 0,$$

$$f \in L^1(0, T; L^2(\Omega)), \quad g \in L^2(0, T; L^2(\partial\Omega)).$$

(7)
Principal Results

Our main result, concerning the stability, is the following:

**Theorem (R. L., A. Mercado, S. Zamorano)**

Suppose that $\Gamma_0 \subset \partial \Omega$ and $T > 0$ satisfy (5)-(6) and the data satisfy (7). Let $M > 0$, and $\alpha_2 \in A_M$ be such that the corresponding solution $u(\alpha_2)$ of (3) (with $\alpha = \alpha_2$) satisfies

$$u(\alpha_2) \in H^3(0, T; L^\infty(\Omega)).$$

Then there exists a constant $C > 0$ such that

$$C^{-1} \left\| \alpha_1 - \alpha_2 \right\|^2_{L^2(\Omega)} \leq \left\| \frac{\partial u(\alpha_1)}{\partial n} - \frac{\partial u(\alpha_2)}{\partial n} \right\|^2_{H^2(0, T; L^2(\Gamma_0))} \leq C \left\| \alpha_1 - \alpha_2 \right\|^2_{L^2(\Omega)}$$

for all $\alpha_1 \in A_M$. 
The hypothesis $u(\alpha_2) \in H^3(0, T; L^\infty(\Omega))$ in Theorem 1 is satisfied if more regularity is imposed on the data.
Remarks

- The hypothesis $u(\alpha_2) \in H^3(0, T; L^\infty(\Omega))$ in Theorem 1 is satisfied if more regularity is imposed on the data.
- The inverse problem studied in this paper was previously considered by Liu and Triggiani. The results obtained in this work requires less regularity.
Remarks

- The hypothesis $u(\alpha_2) \in H^3(0, T; L^\infty(\Omega))$ in Theorem 1 is satisfied if more regularity is imposed on the data.

- The inverse problem studied in this paper was previously considered by Liu and Triggiani.
  The results obtained in this work requires less regularity.

- The hypotheses (5) and (6) on $\Gamma_0$ and $T$ typically arises in the study of stability or observability inequalities for the wave equation.
Remarks

- The hypothesis $u(\alpha_2) \in H^3(0, T; L^\infty(\Omega))$ in Theorem 1 is satisfied if more regularity is imposed on the data.

- The inverse problem studied in this paper was previously considered by Liu and Triggiani. The results obtained in this work requires less regularity.

- The hypotheses (5) and (6) on $\Gamma_0$ and $T$ typically arises in the study of stability or observability inequalities for the wave equation.

- The assumption of the positiveness for $u_2$ appearing in Theorem 1 is classical when applying the Bukhgeim-Klibanov method and Carleman estimates for inverse problems with only one boundary measurement.
Well–posedness

**Theorem ([1], Theorem 2.2)**

Let $b > 0$ and $\alpha \in L^\infty(\Omega)$. Then the solution $u(\alpha)$ is generated by a strongly continuous group on the state space

$$H = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \times L^2(\Omega).$$

That is, for each $(u_0, u_1, u_2) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \times L^2(\Omega)$ and $f \in L^1(0, T; L^2(\Omega))$, there exists a unique solution $U = (u(\alpha), u_t(\alpha), u_{tt}(\alpha)) \in C([0, T]; H)$. 

Kaltenbacher, Barbara and Lasiecka, Irena

Exponential decay for low and higher energies in the third order linear Moore-Gibson-Thompson equation with variable viscosity

The unique solution 
\((u, u_t, u_{tt}) \in C([0, T]; (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \times L^2(\Omega))\) of (3) satisfies
\[
\frac{\partial u}{\partial n} \in H^1(0, T; L^2(\partial \Omega)). \tag{9}
\]

Moreover, the normal derivative satisfies
\[
\left\| \frac{\partial u}{\partial n} \right\|_{H^1(0, T; L^2(\partial \Omega))}^2 \leq C(\|u_0\|_{H^2(\Omega) \cap H^1_0(\Omega)}^2 + \|u_1\|_{H^1_0(\Omega)}^2 + \|u_2\|_{L^2(\Omega)}^2 + \|f\|_{L^1(0, T; L^2(\Omega))}^2). \tag{10}
\]

Consequently, the mapping 
\((f, u_0, u_1, u_2) \mapsto \frac{\partial u}{\partial n}\) is linear continuous from 
\(L^1(0, T; L^2(\Omega)) \times (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \times L^2(\Omega)\) into 
\(H^1(0, T; L^2(\partial \Omega))\).
Carleman estimates

For $x_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ and $\lambda > 0$, we define the weight functions $\phi$ and $\varphi_\lambda$ as follows

$$\varphi_\lambda(x, t) = e^{\lambda \phi(x,t)}, \quad (11)$$

where

$$\phi(x, t) = |x - x_0|^2 - \beta t^2 + M_0, \quad 0 < \beta < 1, \quad (12)$$

and $M_0$ is chosen such that

$$\forall (x, t) \in \Omega \times (-T, T), \quad \phi(x, t) \geq 1. \quad (13)$$
Carleman estimates

**Theorem (R. L., A. Mercado, S. Zamorano)**

Suppose that $\Gamma_0$ and $T$ satisfies (5)–(6). Let $M > 0$ and $\alpha \in A_M$. Let $\beta \in (0, 1)$ such that

$$\beta T > \sup_{x \in \Omega} \|x - x_0\|.$$  

(11)

Then, there exists $s_0 > 0$, $\lambda > 0$ and a positive constant $C$ such that for all $s \geq s_0$

$$\sqrt{s} \int_\Omega e^{2s\varphi_\lambda(0)} |y_{tt}(0)|^2 \, dx$$

$$ + s\lambda c^4 \int_0^T \int_\Omega e^{2s\varphi_\lambda} \varphi_\lambda (|y_t|^2 + |\nabla y|^2) \, dx \, dt + s^3 \lambda^3 c^4 \int_0^T \int_\Omega e^{2s\varphi_\lambda} \varphi_\lambda^3 |y|^2 \, dx \, dt$$

$$ + s\lambda \int_0^T \int_\Omega e^{2s\varphi_\lambda} \varphi_\lambda (|y_{tt}|^2 + |\nabla y_t|^2) \, dx \, dt + s^3 \lambda^3 \int_0^T \int_\Omega e^{2s\varphi_\lambda} \varphi_\lambda^3 |y_t|^2 \, dx \, dt$$

$$\leq C \int_0^T \int_\Omega e^{2s\varphi_\lambda} |f|^2 \, dx \, dt + Cs\lambda \int_0^T \int_{\Gamma_0} e^{2s\varphi_\lambda} \left( |\nabla y_t \cdot n|^2 + c^4 |\nabla y \cdot n|^2 \right) \, d\sigma \, dt,$$

for all $y \in L^2(0, T; H^1_0(\Omega))$ satisfying $f \in L^2(\Omega \times (0, T))$, $y(\cdot, 0) = y_t(\cdot, 0) = 0$ in $\Omega$, and $y_{tt}(\cdot, 0) \in L^2(\Omega)$. 
Sketch of the proof for Carleman estimates

For the wave equation we have

\[ E_w(u) := \sqrt{s} \int_{\Omega} e^{2s\varphi} \lambda(0) |u_t(0)|^2 \, dx + s\lambda \int_{0}^{T} \int_{\Omega} e^{2s\varphi} \lambda \varphi \lambda (|u_t|^2 + |\nabla u|^2) \, dx \, dt + s^3 \lambda^3 \int_{0}^{T} \int_{\Omega} e^{2s\varphi} \lambda \varphi^3 \lambda |u|^2 \, dx \, dt \]

\[ \leq C \int_{0}^{T} \int_{\Omega} e^{2s\varphi} \lambda |L_0 u|^2 \, dx \, dt + Cs \lambda \int_{0}^{T} \int_{\Gamma_0} e^{2s\varphi} \lambda \left( |\nabla u \cdot n|^2 \right) \, d\sigma \, dt, \]

where \( L_0 u = u_{tt} - b\Delta u \) is the classical wave operator.
Sketch of the proof for Carleman estimates

For the wave equation we have

\[
E_w(u) := \sqrt{s} \int_\Omega e^{2s\varphi \lambda(0)} |u_t(0)|^2 \, dx + s\lambda \int_0^T \int_\Omega e^{2s\varphi \lambda} \varphi \lambda (|u_t|^2 + |\nabla u|^2) \, dx \, dt + s^3 \lambda^3 \int_0^T \int_\Omega e^{2s\varphi \lambda} \varphi^3 \lambda |u|^2 \, dx \, dt \\
\leq c \int_0^T \int_\Omega e^{2s\varphi \lambda} |L_0 u|^2 \, dx \, dt + cs\lambda \int_0^T \int_{\Gamma_0} e^{2s\varphi \lambda} \left( |\nabla u \cdot n|^2 \right) \, d\sigma \, dt,
\]

where \( L_0 u = u_{tt} - b\Delta u \) is the classical wave operator. We consider the operator \( L_\alpha u = u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b\Delta u_t \), and we have

\[
L_\alpha u = L_0 u_t + \frac{c^2}{b} L_0 u + (\alpha - \frac{c^2}{b}) u_{tt}.
\]
Sketch of the proof for Carleman estimates

For the wave equation we have

\[ E_w(u) := \sqrt{s} \int_{\Omega} e^{2s\varphi} \chi(0) |u_t(0)|^2 \, dx + s \lambda \int_0^T \int_{\Omega} e^{2s\varphi} \lambda \varphi |u_t|^2 + |\nabla u|^2 \, dx \, dt + s^3 \lambda^3 \int_0^T \int_{\Omega} e^{2s\varphi} \lambda \varphi^3 |u|^2 \, dx \, dt \]

\[ \leq C \int_0^T \int_{\Omega} e^{2s\varphi} |L_0 u|^2 \, dx \, dt + C \lambda \int_0^T \int_{\Gamma_0} e^{2s\varphi} \lambda (|\nabla u \cdot n|^2) \, d\sigma \, dt, \]

where \( L_0 u = u_{tt} - b \Delta u \) is the classical wave operator. We consider the operator \( L_\alpha u = u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t \), and we have

\[ L_\alpha u = L_0 u_t + \frac{c^2}{b} L_0 u + (\alpha - \frac{c^2}{b}) u_{tt}. \]

Now we consider the weight norm \( \| f \|^2_w = \int_0^T \int_{\Omega} e^{2s\varphi} \lambda |f|^2 \, dx \, dt \). And we compute

\[ \| L_\alpha u - (\alpha - \frac{c^2}{b}) u_{tt} \|^2_w = \| L_0 u_t \|^2_w + \frac{c^4}{b^2} \| L_0 u \|^2_w + 2 \frac{c^2}{b} (e^{2s\varphi} \lambda \partial_t L_0 u, L_0 u). \]

Using the Carleman estimates for the wave equation,

\[ C \| L_\alpha u \|^2_w + C \| (\alpha - \frac{c^2}{b}) u_{tt} \|^2_w \geq E_w(u_t) + \frac{c^4}{b^2} E_w(u) + \frac{c^2}{b} \int_0^T \int_{\Omega} e^{2s\varphi} \lambda \partial_t |L_0 u|^2 \, dx \, dt \]
Sketch of the proof for the stability Theorem

Bukhgeim-Klibanov method.
Algorithm to find the coefficient $\alpha$

This algorithm will be based by the work of Baudouin, Buhan and Ervedoza [?], in which they propose a reconstruction algorithm for the potential of the wave equation. Let us consider the following functional

$$J[\mu, f](y) = \frac{1}{2} \int_0^T \int_{\Omega} e^{2s\varphi_{\lambda}} |L_\alpha y - f|^2 \, dx \, dt$$

$$+ \frac{1}{2} \int_0^T \int_{\Gamma_0} e^{2s\varphi_{\lambda}} \left( \left| \frac{\partial y}{\partial n} - \mu \right|^2 + \left| \frac{\partial y_t}{\partial n} - \mu_t \right|^2 \right) \, d\sigma \, dt, \quad (12)$$

where $\alpha \in \mathcal{A}_M$, $g \in L^2(\Omega \times (0, T))$, $\mu \in H^1(0, T; L^2(\Gamma_0))$.


Global Carleman estimates for waves and applications.

Algorithm:

1. **Initialization:** $\alpha^0 = \frac{c^2}{b}$.
2. **Iteration:** From $k$ to $k + 1$

   **Step 1** - Given $\alpha^k$ we consider $\mu^k = \partial_t \left( \frac{\partial u(\alpha^k)}{\partial n} - \frac{\partial u(\alpha)}{\partial n} \right)$ on $\Gamma_0 \times (0, T)$
   where $u(\alpha^k)$ and $u(\alpha)$ are the solution of the problems
   
   $$
   \begin{cases}
   L_{\alpha^k} u = f, \Omega \times (0, T) \\
   u = g, \partial \Omega \times (0, T) \\
   u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, \Omega
   \end{cases}
   $$

   and

   $$
   \begin{cases}
   L_\alpha u = f, \Omega \times (0, T) \\
   u = g, \partial \Omega \times (0, T) \\
   u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2, \Omega
   \end{cases}
   $$

   **Step 2** - Minimize the functional $J[\mu^k, 0]$ on the admissible trajectories $y$.

   **Step 3** - Let $y^{*,k}$ the minimizer of $J[\mu^k, 0]$ and

   $$
   \tilde{\alpha}^{k+1} = \alpha^k + \frac{y_{tt}^{*,k}(\cdot, 0)}{u_2}.
   $$
Algorithm:

1. **Initialization:** \( \alpha^0 = \frac{c^2}{b} \).
2. **Iteration:** From \( k \) to \( k + 1 \)
   - **Step 1** - Given \( \alpha^k \) we consider \( \mu^k = \partial_t \left( \frac{\partial u(\alpha^k)}{\partial n} - \frac{\partial u(\alpha)}{\partial n} \right) \) on \( \Gamma_0 \times (0, T) \)
   - **Step 2** - Minimize the functional \( J[\mu^k, 0] \) on the admissible trajectories \( y \).
   - **Step 3** - Let \( y^{*,k} \) the minimizer of \( J[\mu^k, 0] \) and
     \[
     \tilde{\alpha}^{k+1} = \alpha^k + \frac{y^{*,k} (\cdot, 0)}{u_2}.
     \] (13)
   - **Step 4** - Finally, consider \( \alpha^{k+1} = T(\tilde{\alpha}^{k+1}) \), where
     \[
     T(\alpha) = \begin{cases} 
     M & \text{if } \alpha > M \\
     \alpha & \text{if } \frac{c^2}{b} \leq \alpha \leq M \\
     \frac{c^2}{b} & \text{if } \alpha < \frac{c^2}{b}.
     \end{cases}
     \] (14)

This function \( T \) is to guarantee at each step that \( \alpha^k \) belongs to the admissible set \( A_M \).
The convergence of this algorithm

**Theorem (R. L., A. Mercado, S. Zamorano)**

Assume the same hypotheses of observability Theorem, and the following assumption of $u(\alpha)$:

$$u(\alpha) \in H^3(0, T; L^\infty(\Omega)) \text{ and } |u_2| \geq \eta > 0.$$  \hspace{1cm} (15)

Then, there exists a constant $C > 0$ and $s_0 > 0$ such that for all $s \geq s_0$ and $k \in \mathbb{N}$

$$\int_{\Omega} e^{2s\varphi(0)}(\alpha^{k+1} - \alpha)^2 \, dx \leq \frac{C}{\sqrt{s}} \int_{\Omega} e^{2s\varphi(0)}(\alpha^k - \alpha)^2 \, dx.$$ \hspace{1cm} (16)
Consider a new numerical approach

Baudouin, L. and de Buhan, M. and Ervedoza, S.
Convergent Algorithm Based on Carleman Estimates for the Recovery of a Potential in the Wave Equation.

Thank you for your attention!