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# On some recent issues on the scalar-input bilinear control of PDE's

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**Abstract**

Since the early eighties, it is well-known that the scalar-input bilinear control in infinite dimensions encounters obstuctions. This is closely related to the topological properties of the set of reachable states. It has a dense complement in the usual functional frame. This has first been proved for bilinear control of mechanical systems such as the beam model, and then for the control of quantum systems.

Since then, several positive controllability results were obtained with spatial localization of the control (through a real-valued potential function). These results require as a necessary condition, that an infinite number of real quantities (involving the control, the eigenstates and the targeted eigenstate), are non vanishing. It was also proved that an appropriate asymptotic behavior of these quantities is a sufficient condition for bilinear controllability results to hold. These results mainly concern dispersive equations such as the Schrödinger equation, or hyperbolic equations as the wave or beam equations.

The first purpose of this talk is to present some of the recent results on the bilinear control of first order in time evolution PDE's in an abstract framework, in view of applications to parabolic equations (joint works with P. Cannarsa and C. Urbani). The second purpose is to present a new mathematical methodology to construct explicit infinite sets of potential functions, and provide algorithms to exhibit explicit large classes of control operators that satisfy the necessary and sufficient conditions required for scalar-input controllability to hold. This gives direct concrete results for the bilinear control on heat equations, but also on Schrödinger equation, wave equations, beams equations as well as different examples of boundary conditions (joint work with C. Urbani)

Joint work with:
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# References


[1] Alabau-Boussouira, F.; Cannarsa, P.; Urbani, C., *Superexponential stabilizability of evolution equations of parabolic type via bilinear control*, J. of Evolution Equations **21**, (2021). 941-967.

[2] Alabau-Boussouira, F.; Cannarsa, P.; Urbani, C., *Exact controllability to eigensolutions for evolution equations of parabolic type via bilinear control*, arXiv:2105.05732 (submitted).

[3] Alabau-Boussouira, F.; Urbani, C., *A constructive algorithm for building mixing coupling potentials. Application to bilinear control*, Preprint.

[4] Ball, J.M.; Marsden, J.E.; Slemrod, M., *Controllability for distributed bilinear systems*, SIAM Journal on Control and Optimization **20**, (1982). 575-597.

[5] Beauchard, K.; Coron, J.-M., *Controllability of a quantum particle in a moving potential well*, J. Funct. Anal. **232**, (2006). 328-389.

[6] Beauchard, K.; Laurent, C., *Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control*, Journal Math. Pures Appl. **94**, (2010). 520-554.

[7] Beauchard, K., *Local controllability and non-controllability for a 1d wave equation with bilinear control*, J. Diff. Equat. **250**, (2011). 2064-2098.

[8] Cannarsa, P.; Floridia G., Khapalov A., *Multiplicative controllability for semilinear reaction-diffusion equations with finitely many changes of sign*, J. Math. Pures et Appli. **108**, (2011). 425-458.

[9] Coron, J.-M., *On the small-time local controllability of a quantum particle in a moving one-dimensional infinite square potential well*, Journal Math. Pures Appl. **94**, (2010). 520-554.

[10] Zuazua, E., *Remarks on the controllability of the Schrödinger equation*, Centre de Recherches Mathématiques, CRM Proceedings and Lecture Notes, Quantum Control: Mathematical and Numerical Challenges **33** (2003), 193-211.

[11] Zuazua, E., *Propagation, Observation, and Control of Waves Approximated by Finite Difference Methods*, SIAM Review **47** (2005), 197-243.