Shape and topology optimization of structures built by additive manufacturing

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Outline of the course

- 1 - Introduction: a review of additive manufacturing
- 2 - Parametric optimization and the adjoint method
- 3 - Geometric optimization and Hadamard method
- 4 - Topology optimization and the level set method
- 5 - Typical constraints from additive manufacturing
- 6 - Optimization of lattice materials
- 7 - Coupled shape and laser path optimization

A "hot" topic with a lot of room for new ideas and modeling...

Chapter 3 - Geometric optimization and Hadamard method

- I - Introduction and setting
- II - Shape parametrization: Hadamard method
- III - Example of shape derivatives
- IV - Numerical algorithm and results

G. Allaire, *Conception optimale de structures*, Mathématiques et Applications, Vol. 58, Springer (2007).

G. Allaire, C. Dapogny, F. Jouve, *Shape and topology optimization*, in Geometric partial differential equations, part II, A. Bonito and R. Nochetto eds., pp.1-132, Handbook of Numerical Analysis, vol. 22, Elsevier (2021).

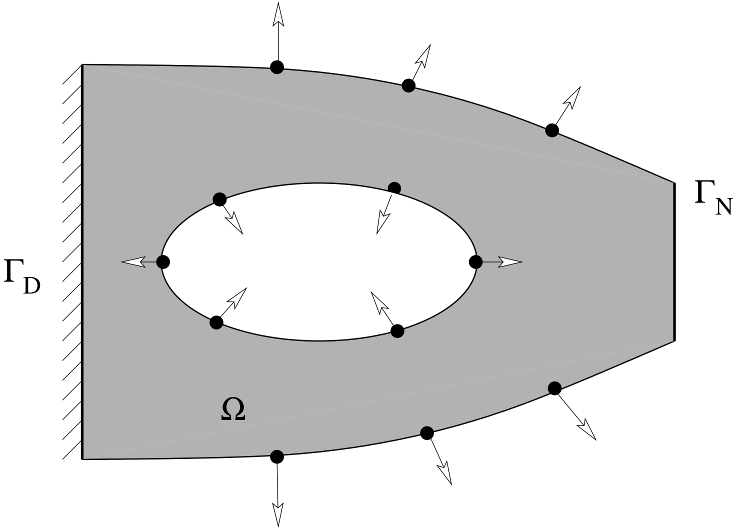
Model problem: a membrane is occupying a **variable** domain Ω in $\mathbb{R}^N$ with boundary

$$\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D,$$

where $\Gamma \neq \emptyset$ is the variable part of the boundary, $\Gamma_D \neq \emptyset$ is a fixed part of the boundary where the membrane is clamped, and $\Gamma_N \neq \emptyset$ is another fixed part of the boundary where the loads $g \in L^2(\Gamma_N)$ are applied.

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma \end{cases}$$

(No bulk forces to simplify)



Γ_D
Γ_N
Ω

Geometric shape optimization problem

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega)$$

We must defined the set of admissible shapes $\mathcal{U}_{ad}$. That is the main difficulty.

Examples:

- Compliance or work done by the load (rigidity measure)

$$J(\Omega) = \int_{\Gamma_N} gu \, ds$$

- Least square criterion for a target displacement $u_0 \in L^2(\Omega)$

$$J(\Omega) = \int_{\Omega} |u - u_0|^2 dx$$

where u depends on Ω through the state equation.

Existence of optimal shapes

In full generality, there does not exist any optimal shape !

- Existence under a geometric constraint.
- Existence under a topological constraint.
- Existence under a regularity constraint.
- Counter-example in the absence of these conditions.

- How to represent or parametrize shapes ?
- How to define the set $\mathcal{U}_{ad}$ of admissible shapes ?
- How to compute gradients with respect to shapes ?

One possible approach: fix a Lipschitz reference domain Ω_0 and consider only deformations of Ω_0 by diffeomorphisms T.

A space of diffeomorphisms (or smooth one-to-one map) in $\mathbb{R}^N$

$$\mathcal{T} = \left\{ T \text{ such that } (T - \mathrm{Id}) \text{ and } (T^{-1} - \mathrm{Id}) \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N) \right\}.$$

(They are perturbations of the identity Id: $x \to x$.)

Definition of $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$. Space of Lipschitz vectors fields:

$$\phi : \begin{cases} \mathbb{R}^N & \to & \mathbb{R}^N \\ x & \to & \phi(x) \end{cases}$$

$$\|\phi\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N} \left(|\phi(x)|_{\mathbb{R}^N} + |\nabla\phi(x)|_{\mathbb{R}^{N\times N}}\right) < \infty$$

Define a space of admissible shapes as

$$\mathcal{U}_{ad}(\Omega_0) = \{\Omega \text{ such that there exists } T \in \mathcal{T}, \Omega = T(\Omega_0)\}\,.$$

- Each shape Ω is parametrized by a diffeomorphism T (not unique !).
- All admissible shapes have the same topology.

Hadamard method

Recall that the considered diffeomorphisms T are perturbations of the identity.
Therefore, we restrict ourselves to diffeomorphisms of the type

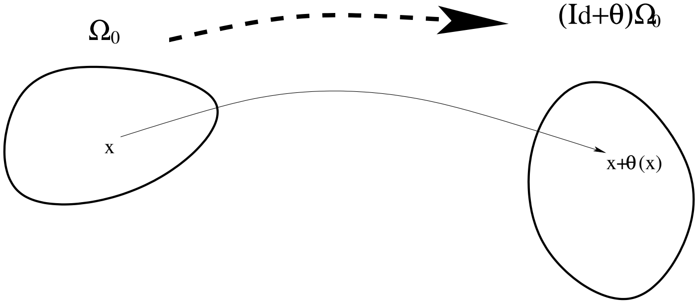
$$T = \mathrm{Id} + \theta \quad \text{with} \quad \theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$$

Idea: we differentiate $\theta \to J\big((\mathrm{Id} + \theta)\Omega_0\big)$ at 0.

Remark. This approach generalizes the Hadamard method of boundary shape variations along the normal: $\Omega_0 \to \Omega_t$ for $t \geq 0$

$$\partial\Omega_t = \Big\{ x_t \in \mathbb{R}^N \mid \exists x_0 \in \partial\Omega_0 \mid x_t = x_0 + t\, g(x_0)\, n(x_0) \Big\}$$

with a given incremental function g.



The shape $\Omega = (\,\mathrm{Id} + \theta)(\Omega_0)$ is defined by

$$\Omega = \{x + \theta(x) \mid x \in \Omega_0\}\,.$$

Thus $\theta(x)$ is a vector field which plays the role of the **displacement** of the reference domain Ω_0.

Lemma. For any $\theta \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ satisfying $\|\theta\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)} < 1$, the map $T = \mathrm{Id} + \theta$ is one-to-one into $\mathbb{R}^N$ and belongs to the set $\mathcal{T}$.

Proof. Exercise !

Definition. Let $J(\Omega)$ be a map from the set of admissible shapes $\mathcal{U}_{ad}(\Omega_0)$ into $\mathbb{R}$. We say that J is shape differentiable at Ω_0 if the function

$$\theta \to J\big((\mathrm{Id} + \theta)(\Omega_0)\big)$$

is Fréchet differentiable at 0 in the Banach space $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$, i.e., there exists a linear continuous form $L = J'(\Omega_0)$ on $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ such that

$$J\big((\mathrm{Id}+\theta)(\Omega_0)\big) = J(\Omega_0) + L(\theta) + o(\theta) \quad , \quad \text{with} \quad \lim_{\theta\to 0} \frac{|o(\theta)|}{\|\theta\|} = 0 \, .$$

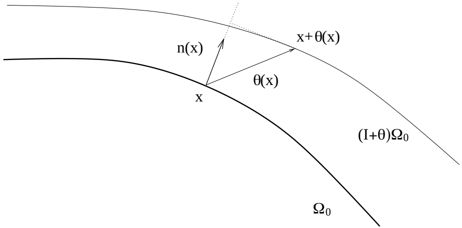
$J'(\Omega_0)$ is called the shape derivative and $J'(\Omega_0)(\theta)$ is a directional derivative.

Proposition. Let Ω_0 be a smooth bounded open set of $\mathbb{R}^N$. Let J be a differentiable map at Ω_0 from $\mathcal{U}_{ad}(\Omega_0)$ into $\mathbb{R}$. Its directional derivative $J'(\Omega_0)(\theta)$ depends only on the normal trace on the boundary of θ, i.e.

$$J'(\Omega_0)(\theta_1) = J'(\Omega_0)(\theta_2)$$

if $\theta_1, \theta_2 \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ satisfy

$$\theta_1 \cdot n = \theta_2 \cdot n \quad \text{on } \partial\Omega_0.$$



Proposition (loose statement). Let Ω_0 be a smooth bounded open set of $\mathbb{R}^N$. Let J be a differentiable map at Ω_0 from $\mathcal{U}_{ad}(\Omega_0)$ into $\mathbb{R}$. Its directional derivative can be written

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} j(x)\,\theta\cdot n\,ds$$

for some distribution j on $\partial\Omega_0$.

Lemma. Let Ω_0 be a smooth bounded open set of $\mathbb{R}^N$, $f(x) \in W^{1,1}(\mathbb{R}^N)$ and J the map from $\mathcal{U}_{ad}(\Omega_0)$ into $\mathbb{R}$ defined by

$$J(\Omega) = \int_\Omega f(x)\, dx.$$

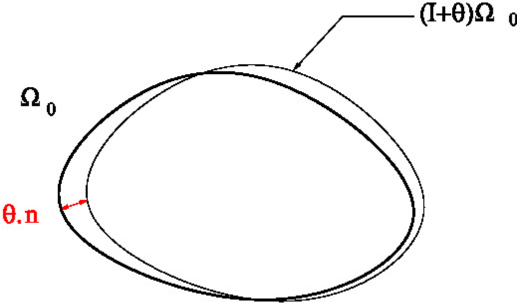
Then J is shape differentiable at Ω_0 and

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} \mathrm{div}\big(\theta(x)\, f(x)\big)\, dx = \int_{\partial\Omega_0} \theta(x)\cdot n(x)\, f(x)\, ds$$

for any $\theta \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$.

Remark. To prove the lemma, the safest way (but not the easiest) is to make a change of variables to get back to the reference domain Ω_0.

Intuitive proof



Surface swept by the transformation: difference between $(\mathrm{Id}+\theta)\Omega_0$ and $\Omega_0 \approx \partial\Omega_0 \times \Big(\theta \cdot n\Big)$. Thus

$$\int_{(\mathrm{Id}+\theta)\Omega_0} f(x)\,dx = \int_{\Omega_0} f(x)\,dx + \int_{\partial\Omega_0} f(x)\theta \cdot n\,ds + o(\theta).$$

Derivative of a surface integral

Lemma. Let Ω_0 be a smooth bounded open set of $\mathbb{R}^N$, $f(x) \in W^{2,1}(\mathbb{R}^N)$ and J the map from $\mathcal{C}(\Omega_0)$ into $\mathbb{R}$ defined by

$$J(\Omega) = \int_{\partial\Omega} f(x)\, ds.$$

Then J is shape differentiable at Ω_0 and

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \left(\nabla f \cdot \theta + f\left(\mathrm{div}\theta - \nabla\theta n \cdot n\right)\right) ds$$

for any $\theta \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$. By a (boundary) integration by parts this formula is equivalent to

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \left(\frac{\partial f}{\partial n} + Hf\right) ds,$$

where H is the mean curvature of $\partial\Omega_0$ defined by $H = \mathrm{div}\, n$.

This is tricky ! Let $u(\Omega, x)$ be a function defined on the domain Ω.
There exist two notions of derivative:

1) Eulerian (or shape) derivative U

$$u((\mathrm{Id} + \theta)\Omega_0, x) = u(\Omega_0, x) + U(\theta, x) + o(\theta)$$

OK if $x \in \Omega_0 \cap (\mathrm{Id} + \theta)\Omega_0$ (makes no sense on the boundary).

2) Lagrangian (or material) derivative Y

We define the **transported** function $\overline{u}(\theta)$ on Ω_0 by

$$\overline{u}(\theta, x) = u \circ (\mathrm{Id} + \theta) = u\Big((\mathrm{Id} + \theta)\Omega_0, x + \theta(x)\Big) \quad \forall\, x \in \Omega_0.$$

The Lagrangian derivative Y is obtained by differentiating $\overline{u}(\theta, x)$

$$\overline{u}(\theta, x) = \overline{u}(0, x) + Y(\theta, x) + o(\theta)$$

This is more rigorous and less prone to errors.

Fast derivation: the Lagrangian method

- To prove that the solution of a p.d.e. is shape differentiable and to find the formula for its derivative is always tedious.
- Instead, J. Céa proposed a simpler and faster (albeit formal) method, called the Lagrangian method that we use in the sequel.
- The Lagrangian allows us to find the correct definition of the adjoint state too.
- The Lagrangian yields the shape derivative of an objective function without knowing the shape derivative of the solution of the p.d.e.
- It is easy for Neumann boundary conditions, a little more involved for Dirichlet ones.

Neumann problem

We consider the shape optimization problem

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega) = \int_{\Omega} j(u) dx$$

with $u \equiv u(\Omega)$ the solution in $H^1(\Omega)$ of

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega, \end{cases}$$

with $f \in H^1(\mathbb{R}^N)$ and $g \in H^2(\mathbb{R}^N)$.
The variational formulation is: find $u \in H^1(\Omega)$ such that, for any $\phi \in H^1(\Omega)$,

$$\int_{\Omega} \Big(\nabla u \cdot \nabla \phi + u\phi \Big) dx = \int_{\Omega} f \phi \, dx + \int_{\partial\Omega} g\phi \, ds$$

Fast derivation for Neumann boundary conditions

Crucial point: if Ω is Lipschitz, then

$$H^1(\Omega) = \left\{ \phi_{|\Omega} \text{ with } \phi \in H^1(\mathbb{R}^N) \right\}$$

The Lagrangian is defined as the sum of J and of the variational formulation of the state equation

$$\mathcal{L}(\Omega, \hat{u}, \hat{p}) = \int_\Omega j(\hat{u})\, dx + \int_\Omega \Big(\nabla \hat{u} \cdot \nabla \hat{p} + \hat{u}\hat{p} - f\,\hat{p} \Big) dx - \int_{\partial\Omega} g\,\hat{p}\, ds,$$

with $\hat{u}$ and $\hat{p} \in H^1(\mathbb{R}^N)$.

It is important to notice that the space $H^1(\mathbb{R}^N)$ does not depend on Ω and thus the three variables in $\mathcal{L}$ are clearly independent.

We compute the three partial derivatives of the Lagrangian $\mathcal{L}(\Omega, \hat{u}, \hat{p})$.

The partial derivative of $\mathcal{L}$ with respect to p in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial p}(\Omega, \hat{u}, \hat{p}), \phi \rangle = \int_\Omega \Big(\nabla \hat{u} \cdot \nabla \phi + \hat{u}\phi - f\phi \Big) dx - \int_{\partial\Omega} g\phi \, ds,$$

which, equal to 0, gives the variational formulation of the state u.

The partial derivative of $\mathcal{L}$ with respect to u in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial u}(\Omega, \hat{u}, \hat{p}), \phi \rangle = \int_\Omega j'(\hat{u})\phi \, dx + \int_\Omega \Big(\nabla \phi \cdot \nabla \hat{p} + \phi\hat{p} \Big) dx,$$

which, equal to 0, gives the variational formulation of the adjoint p.

The partial derivative of $\mathcal{L}$ with respect to Ω in the direction θ is

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, \hat{u}, \hat{p})(\theta) = \int_{\partial\Omega} \theta\cdot n \left(j(\hat{u}) + \nabla \hat{u} \cdot \nabla \hat{p} + \hat{u}\hat{p} - f\hat{p} - \frac{\partial (g\hat{p})}{\partial n} - Hg\hat{p} \right) ds.$$

When evaluating this derivative with the state $u(\Omega)$ and the adjoint $p(\Omega)$, we precisely find the derivative of the objective function

Proposition. The shape derivative of the objective function is

$$J'(\Omega)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}\Big(\Omega, u(\Omega), p(\Omega)\Big)(\theta)$$

Proof of the proposition

Proof. Indeed, if we differentiate the equality

$$\mathcal{L}(\Omega, u(\Omega), \hat{p}) = J(\Omega) \qquad \forall\, \hat{p} \in H^1(\mathbb{R}^N),$$

the chain rule lemma yields

$$J'(\Omega)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, u(\Omega), \hat{p})(\theta) + \langle \frac{\partial \mathcal{L}}{\partial u}(\Omega, u(\Omega), \hat{p}), u'(\Omega)(\theta) \rangle$$

Taking $\hat{p} = p(\Omega)$, the last term cancels since $p(\Omega)$ is the solution of the adjoint equation.

Thanks to this computation, the "correct" result can be guessed for $J'(\Omega)$ without using the notions of shape or material derivatives.

Nevertheless, this "fast" computation of the shape derivative $J'(\Omega)$ is rigorously valid only if u is known to be shape differentiable.

Fast derivation for Dirichlet boundary conditions

It is more involved ! Let $u \in H^1_0(\Omega)$ be the solution of

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \phi \in H^1_0(\Omega).$$

The "usual" Lagrangian is

$$\mathcal{L}(\Omega, \hat{u}, \hat{p}) = \int_\Omega j(\hat{u}) \, dx + \int_\Omega \Big(\nabla \hat{u} \cdot \nabla \hat{p} - f \hat{p} \Big) dx,$$

for $\hat{u}, \hat{p} \in H^1_0(\Omega)$. The variables $(\Omega, \hat{u}, \hat{p})$ are not independent !

Indeed, the functions $\hat{u}$ and $\hat{p}$ satisfy

$$\hat{u} \;=\; \hat{p} \;=\; 0 \quad \text{on } \partial\Omega.$$

Another Lagrangian has to be introduced.

The Dirichlet boundary condition is penalized

$$\mathcal{L}(\Omega, \hat{u}, \hat{p}, \lambda) = \int_\Omega j(\hat{u})\, dx - \int_\Omega (\Delta \hat{u} + f)\hat{p}\, dx + \int_{\partial\Omega} \lambda \hat{u}\, ds$$

where λ is the Lagrange multiplier for the boundary condition. It is now possible to differentiate since the 4 variables $\hat{u}, \hat{p}, \lambda \in H^1(\mathbb{R}^N)$ and Ω are independent.

Of course, we recover

$$\sup_{\hat{p},\lambda} \mathcal{L}(\Omega, \hat{u}, \hat{p}, \lambda) = \begin{cases} \displaystyle\int_\Omega j(u)\, dx = J(\Omega) & \text{if } \hat{u} \equiv u, \\ +\infty & \text{otherwise.} \end{cases}$$

By definition of the Lagrangian:

1) the partial derivative of $\mathcal{L}$ with respect to p in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial p}(\Omega, \hat{u}, \hat{p}, \lambda), \phi \rangle = -\int_\Omega \phi \Big(\Delta \hat{u} + f \Big) dx,$$

which, equal to 0, gives the state equation,

2) the partial derivative of $\mathcal{L}$ with respect to λ in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial \lambda}(\Omega, \hat{u}, \hat{p}, \lambda), \phi \rangle = \int_{\partial\Omega} \phi \hat{u} \, dx,$$

which, equal to 0, gives the Dirichlet boundary condition for the state equation.

3) To compute the partial derivative of $\mathcal{L}$ with respect to u, we perform a first integration by parts

$$\mathcal{L}(\Omega, \hat{u}, \hat{p}, \lambda) = \int_\Omega j(\hat{u})\, dx + \int_\Omega (\nabla \hat{u} \cdot \nabla \hat{p} - f\hat{p})\, dx + \int_{\partial\Omega} \left(\lambda \hat{u} - \frac{\partial \hat{u}}{\partial n} \hat{p} \right) ds,$$

then a second integration by parts

$$\mathcal{L}(\Omega, \hat{u}, \hat{p}, \lambda) = \int_\Omega j(\hat{u})\, dx - \int_\Omega (\hat{u}\Delta \hat{p} - f\hat{p})\, dx + \int_{\partial\Omega} \left(\lambda \hat{u} - \frac{\partial \hat{u}}{\partial n} \hat{p} + \frac{\partial \hat{p}}{\partial n} \hat{u} \right) ds.$$

We now can differentiate in the direction $\phi \in H^1(\mathbb{R}^N)$

$$\langle \frac{\partial \mathcal{L}}{\partial u}(\Omega, \hat{u}, \hat{p}), \phi \rangle = \int_\Omega j'(\hat{u})\phi\, dx - \int_\Omega \phi \Delta \hat{p}\, dx + \int_{\partial\Omega} \left(-\hat{p} \frac{\partial \phi}{\partial n} + \phi \left(\lambda + \frac{\partial \hat{p}}{\partial n} \right) \right) ds$$

which, equal to 0, gives three relationships, the two first ones being the adjoint problem.

Lagrangian for Dirichlet boundary conditions (end)

1. If ϕ has compact support in Ω, we get

$$-\Delta p = -j'(u) \quad \text{in} \quad \Omega.$$

2. If $\phi = 0$ on $\partial\Omega$ with any value of $\frac{\partial \phi}{\partial n}$ in $L^2(\partial\Omega)$, we deduce

$$p = 0 \quad \text{on} \quad \partial\Omega.$$

3. If ϕ is now varying in the full $H^1(\Omega)$, we find

$$\frac{\partial p}{\partial n} + \lambda = 0 \quad \text{on} \quad \partial\Omega.$$

The adjoint problem has actually been recovered but furthermore the optimal Lagrange multiplier λ has been characterized.

4) Eventually, the shape partial derivative is

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, \hat{u}, \hat{p}, \lambda)(\theta) = \int_{\partial \Omega} \theta \cdot n \Big(j(\hat{u}) - (\Delta \hat{u} + f)\hat{p} + \frac{\partial (\hat{u}\lambda)}{\partial n} + H \hat{u} \lambda \Big) ds$$

Proposition. The shape derivative of the objective function is

$$J'(\Omega)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}\Big(\Omega, u(\Omega), p(\Omega), \lambda\Big)(\theta)$$

Knowing that $u = p = 0$ on $\partial\Omega$ and $\lambda = -\frac{\partial p}{\partial n}$ we deduce

$$J'(\Omega)(\theta) = \int_{\partial \Omega} \theta \cdot n \Big(j(0) - \frac{\partial u}{\partial n}\frac{\partial p}{\partial n} \Big) ds$$

Proof of the proposition

Proof. We differentiate the equality

$$\mathcal{L}(\Omega, u(\Omega), \hat{p}, \lambda) = J(\Omega) \qquad \forall\, \hat{p}, \lambda \in H^1(\mathbb{R}^N),$$

the chain rule lemma yields

$$J'(\Omega)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, u(\Omega), \hat{p}, \lambda)(\theta) + \langle \frac{\partial \mathcal{L}}{\partial u}(\Omega, u(\Omega), \hat{p}, \lambda), u'(\Omega)(\theta) \rangle$$

Taking $\hat{p} = p(\Omega)$ and λ optimal, the last term cancels since $p(\Omega)$ is the solution of the adjoint equation and $\frac{\partial p}{\partial n} + \lambda = 0$.

Again, this "fast" computation of the shape derivative $J'(\Omega)$ is rigorously valid only if u is known to be shape differentiable.

Let $t > 0$ be a given descent step. We compute a sequence $\Omega_k \in \mathcal{U}_{ad}$ by

1. Initialization of the shape Ω_0.
2. Iterations until convergence, for $k \geq 0$:

$$\Omega_{k+1} = (\mathrm{Id} + \theta_k)\Omega_k \quad \text{with} \quad \theta_k = t(j_k - \ell_k)n,$$

where n is the normal to the boundary $\partial\Omega_k$ and $\ell_k \in \mathbb{R}$ is the Lagrange multiplier such that Ω_{k+1} satisfies the volume constraint. The shape derivative is given on the boundary Γ_k by

$$J'(\Omega_k)(\theta) = -\int_\Gamma \theta \cdot n\, j_k\, ds$$

Free boundary Γ. Fixed boundary Γ_N and Γ_D.

$$\begin{cases} -\operatorname{div}\sigma = 0 & \text{in } \Omega \\ \sigma = 2\mu e(u) + \lambda \operatorname{tr}(e(u))\,\mathrm{Id} & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \sigma n = g & \text{on } \Gamma_N \\ \sigma n = 0 & \text{on } \Gamma, \end{cases}$$

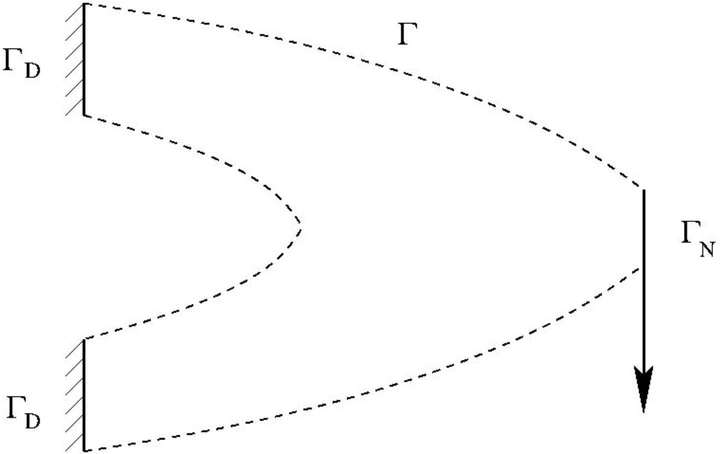
with $e(u) = (\nabla u + (\nabla u)^t)/2$. Compliance is minimized

$$J(\Omega) = \int_{\Gamma_N} g \cdot u \, dx.$$

In such a (self-adjoint) case we get

$$J'(\Omega)(\theta) = -\int_\Gamma \theta \cdot n \left(2\mu |e(u)|^2 + \lambda (\operatorname{tr} e(u))^2\right) ds.$$

Boundary conditions for an elastic cantilever: Γ_D is the left vertical side, Γ_N is the right vertical side, and Γ (dashed line) is the remaining boundary.



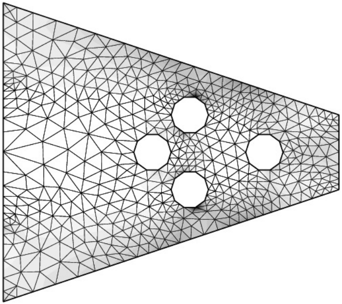
Mesh deformation

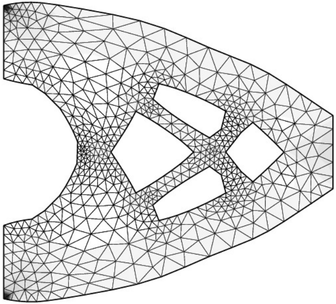
To change the shape we need to automatically remesh the new shape, or at least to deform the mesh at each iteration.

- Displacement field θ proportional to n (normal to the boundary), merely defined on the boundary.
- In such a case we have to extend θ inside the shape.
- We need to check that the displaced boundaries do not cross...
- Nevertheless, in case of large shape deformations we must remesh (it is computationally costly).
- Often the algorithm stops before convergence because of geometrical constraints.

FreeFem++ computations ; scripts available on the web page
`http://www.cmap.polytechnique.fr/~allaire/cours_X_annee3.html`

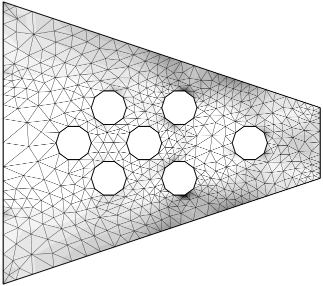
Numerical example for the cantilever:
initial shape (left), “optimal” shape (right)

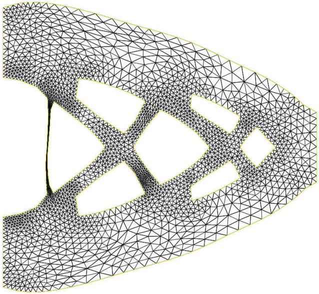




- Convergence in 20 iterations.
- Global or local minimum ?
- No topology changes.

Numerical example for the cantilever:
initial shape (left), “optimal” shape (right)





- No convergence ! Rather, problem with a thin bar...
- One more local minimum !