Shape and topology optimization of structures built by additive manufacturing

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Outline of the course

- 1 - Introduction: a review of additive manufacturing
- 2 - Parametric optimization and the adjoint method
- 3 - Geometric optimization and Hadamard method
- 4 - Topology optimization and the level set method
- 5 - Typical constraints from additive manufacturing
- 6 - Optimization of lattice materials
- 7 - Coupled shape and laser path optimization

A "hot" topic with a lot of room for new ideas and modeling...

Chapter 4 - Topology optimization and the level set method

- I - Introduction and motivation
- II - Level set method
- III - Application to topology optimization
- IV - Numerical algorithm and results

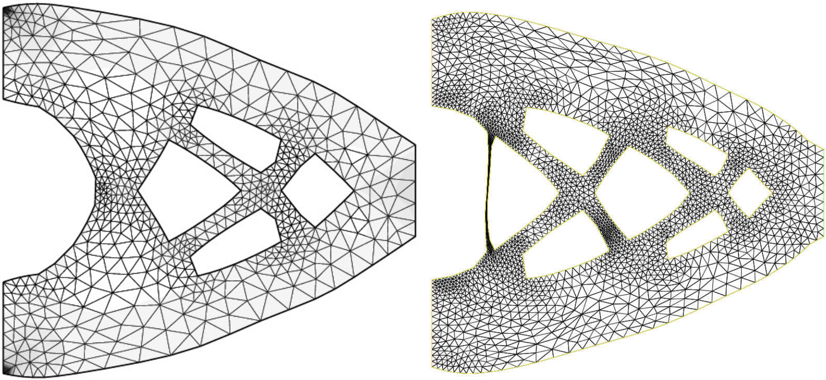
G. Allaire, C. Dapogny, F. Jouve, *Shape and topology optimization*, in Geometric partial differential equations, part II, A. Bonito and R. Nochetto eds., pp.1-132, Handbook of Numerical Analysis, vol. 22, Elsevier (2021).

I - Introduction and motivation

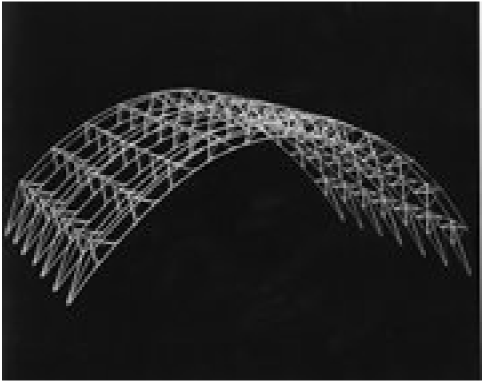
- Geometric optimization **cannot** change the topology.
- Many local minimizers are due to topological constraints.
- Sometimes, geometric optimization want to change the topology but it cannot in practice.
- This issue is both theoretical and numerical.

Topology optimization is required !

Different optimal shapes with different topologies



The art of structure is where to put the holes



Robert Le Ricolais, architect and engineer, 1894-1977

Airbus A380 wing

Airbus A380 wing

- Homogenization method (the oldest one).
- SIMP, Solid Isotropic Material with Penalization (the most popular one).
- Topological gradient or asymptotics.
- Phase field method.
- Level set method.

We focus exclusively on the last one.

II - Level set method

The level set method was introduced by Osher and Sethian (JCP 1988).

Applications in structural optimization:

- Early works: Sethian and Wiegmann (JCP 2000), Osher and Santosa (JCP 2001).
- A.-Jouve-Toader (CRAS 2002, JCP 2004), M. Wang et al. (CMAME 2003).
- Similar works with a phase field model: Bourdin and Chambolle (COCV 2003).
- Many, many works since then !

Front propagation by the level set method

More general problem: how to move a surface $x(t)$ according to a given velocity $\vec{v}(t,x)$.

Lagrangian approach: let us solve o.d.e.'s

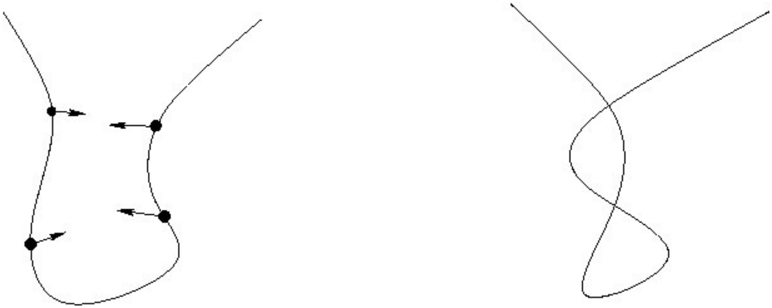
$$\begin{cases} \frac{dx}{dt} = \vec{v}(t, x(t)) \\ x(0) = x_0 \end{cases}$$

Surface evolution:

$$\Gamma(0) = \{x_0\} \quad \Rightarrow \quad \Gamma(t) = \{x(t)\}$$

- Reversible method: to go back in time, change the velocity sign !
- **Shape tracking method**.

Issue with Lagrangian methods



- Problems with self-intersection and singularity !
- How to handle a velocity $\vec{v}$ which depends on the surface through its normal, mean curvature, etc. ?
- How to devise **an Eulerian approach** ?
- It is necessary to make the evolution irreversible.

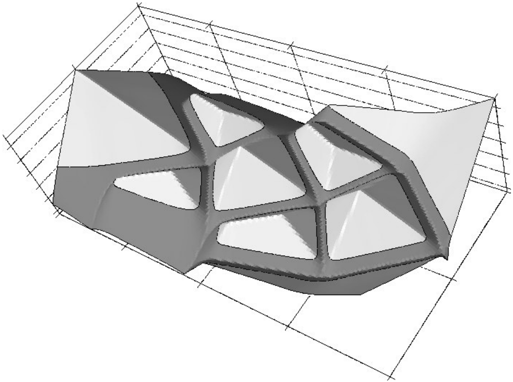
Shape capturing method on a fixed mesh of a “large” box D.

A shape Ω is parametrized by a level set function

$$\begin{cases} \psi(x) = 0 & \Leftrightarrow x \in \partial\Omega \cap D \\ \psi(x) < 0 & \Leftrightarrow x \in \Omega \\ \psi(x) > 0 & \Leftrightarrow x \in (D \setminus \Omega) \end{cases}$$

The normal n to Ω is given by $\nabla\psi/|\nabla\psi|$ and the mean curvature H is the divergence of n. These formulas make sense everywhere in D on not only on the boundary $\partial\Omega$.

Example of a level set function



Hamilton Jacobi equation

Assume that a shape $\Omega(t)$ evolves with a (scalar) normal velocity $V(t,x)$. Then

$$\psi\Big(t,x(t)\Big)=0 \quad \text{for any } x(t)\in\partial\Omega(t).$$

Deriving in t yields

$$\frac{\partial\psi}{\partial t}+\dot{x}(t)\cdot\nabla_x\psi=\frac{\partial\psi}{\partial t}+Vn\cdot\nabla_x\psi=0.$$

(The same is true for any level set $\psi\Big(t,x(t)\Big)=C$.)

Since $n=\nabla_x\psi/|\nabla_x\psi|$ we obtain

$$\frac{\partial\psi}{\partial t}+V|\nabla_x\psi|=0.$$

This Hamilton Jacobi equation is posed in the whole box D, and not only on the boundary $\partial\Omega$, if the velocity V is known everywhere.

Invariance with respect to extension outside Γ

The only meaningful information is the level set $\psi(t) = 0$.
It should not depend on the choice of extended initial data ψ_0 such that $\Gamma(0) = \{\psi_0 = 0\}$.

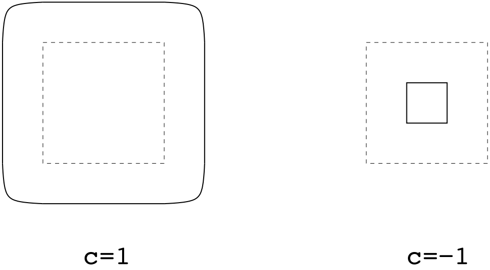
Lemma. Let $z \to h(z)$ be an increasing function such that $h(0) = 0$. If ψ is a H-J solution for the initial data ψ_0, then $h(\psi)$ is a solution for $h(\psi_0)$ too.

Formal proof. Multiply the H-J equation by $h'(\psi) \geq 0$ which can be put inside the absolute values.

Consequence: the level set $h(\psi)(t) = 0$ is the same whatever the choice of the function h.

Cf. works of Barles, Chen-Giga-Goto, Evans-Spruck.

Example of an explicit solution



$$\frac{\partial \psi}{\partial t} + c|\nabla \psi| = 0.$$

A viscosity solution is $\boxed{\psi(t,x) = d(x,\Gamma_0) - c\,t}$ with $d(x,\Gamma_0)$ the signed distance to the initial surface. Irreversible solution !

Conclusion: some corners remain corners, others get rounded !
Numerical schemes must preserve this property (upwinding).

Idea of the method: combine the level set algorithm with the shape derivative of Hadamard.

Recall that a shape derivative is

$$J'(\Omega)(\theta) = \int_{\partial\Omega} j(u,p)\,\theta\cdot n\,ds.$$

Gradient algorithm: choose θ such that $J'(\Omega)(\theta) < 0$ and move the shape

$$\Omega_t = \Big(\mathrm{Id} + t\theta\Big)\Omega \quad \text{with} \quad \theta = -j(u,p)\,n$$

for some descent step $t > 0$.

Clearly, θ is a normal speed and the descent step is like a pseudo-time.

Idea of the level set method

Instead of moving the mesh of Ω, we advect the level set function ψ of Ω

$$\frac{\partial \psi}{\partial t} - j|\nabla_x \psi| = 0 \quad \text{in } D$$

where the "pseudo-time" t is the descent step.

- It requires to embed Ω in a "large" box D.
- The advection velocity $V = -j$ is given by a shape derivative.
- This advection velocity must be extended to the whole D from its knowledge on $\partial\Omega$.
- The whole box D must be meshed and not only the shape Ω.

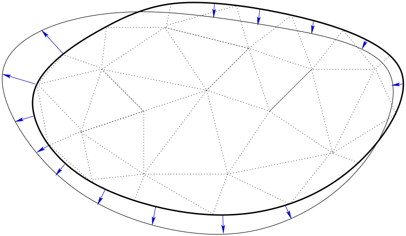
Shape capturing algorithm: the mesh of D is fixed and Ω (which is varying during optimization) is never meshed explicitly.

The state u and the adjoint p are computed in D and not Ω. The void region $D \setminus \Omega$ is filled with a weak ersatz material.

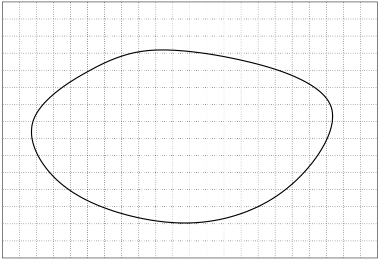
$$\begin{cases} -\operatorname{div}\left(A^* e(u)\right) = 0 & \text{in } D \\ u = 0 & \text{on } \Gamma_D \\ \left(A^* e(u)\right) n = g & \text{on } \Gamma_N \\ \left(A^* e(u)\right) n = 0 & \text{on } \partial D \setminus (\Gamma_N \cup \Gamma_D). \end{cases}$$

with an elasticity tensor A^* defined as A inside Ω and εA inside $D \setminus \Omega$ with $\varepsilon \approx 10^{-4}$.

Tracking versus shape capturing



Shape tracking



Shape capturing

Two algorithms:

1. On a cartesian mesh use an explicit upwind finite difference scheme of order 2:
   Sethian, *Level Set Methods and Fast Marching Methods*, Cambridge University Press (1999).
2. On an unstructured mesh use the methods of characteristics:
   Bui, Dapogny, Frey, *International Journal for Numerical Methods in Fluids* 70 (7), 899-922 (2012).

We have to solve the Hamilton-Jacobi equation

$$\frac{\partial \psi}{\partial t} + V|\nabla_x \psi| = 0 \quad \text{in } D$$

with a normal velocity V that decreases the objective function.

$$J'(\Omega)(\theta) = \int_\Gamma j\, \theta \cdot n\, ds$$

Is the choice $V = \theta \cdot n = -j$ the only one ?

No !

We can extend and regularize j to obtain another V which is still a descent direction, meaning that for $\theta = V\, n$

$$J'(\Omega)(\theta) \leq 0$$

$$J'(\Omega)(\theta) = \int_\Gamma j\,\theta \cdot n\,ds$$

We extend and regularize $-j$ by solving

$$\begin{cases} -\Delta V = -j\delta_\Gamma & \text{in } D \\ V = 0 & \text{on } \Gamma_D \cup \Gamma_N \\ \frac{\partial V}{\partial n} = 0 & \text{on } \partial D \setminus (\Gamma_D \cup \Gamma_N) \end{cases}$$

where δ_Γ is the Dirac mass carried by Γ. The variational formulation is

$$\int_D \nabla V \cdot \nabla \varphi\,dx = -\int_\Gamma j\,\varphi\,ds \qquad \forall\,\varphi$$

The velocity V is smoother than $-j$ and it is a descent direction because for $\theta = V\,n$

$$J'(\Omega)(\theta) = \int_\Gamma j\,V\,ds = -\int_D |\nabla V|^2 dx \le 0$$

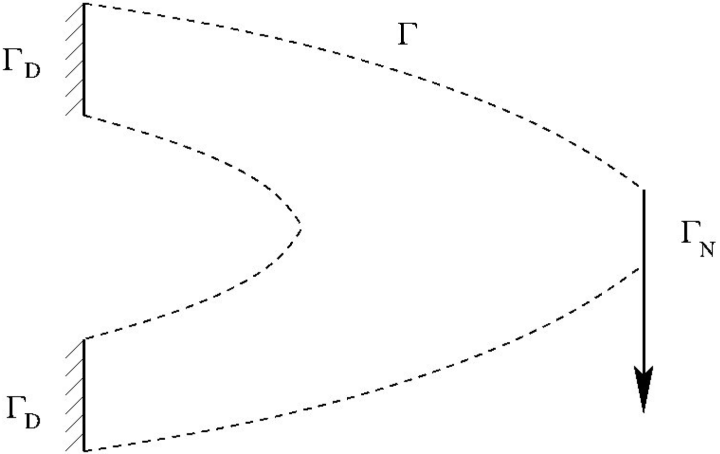
1. Initialization of the level set function ψ_0 (including holes).
2. Iteration until convergence for $k \geq 1$:
    1. Computation of u_k and p_k by solving **linearized elasticity problem** with the shape ψ_k. Evaluation of the shape gradient = normal velocity V_k
    2. Transport of the shape by V_k **(Hamilton Jacobi equation)** to obtain a new shape ψ_{k+1}.

Algorithmic issues

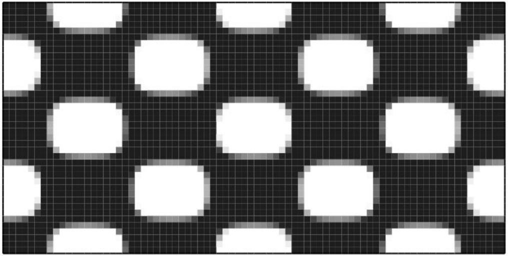
- Elasticity tensor A^* defined as a "mixture" of A and a weak ersatz material mimicking holes: $A^* = \rho A$ with $10^{-3} \leq \rho \leq 1$ and $\rho =$ volume of the shape $\psi < 0$ in each cell.
- At each elasticity analysis, we perform many time steps of transport (its number is controlled by the decrease of the objective function).
- Occasionally, re-initialization of the level set function ψ as the signed distance to the interface.

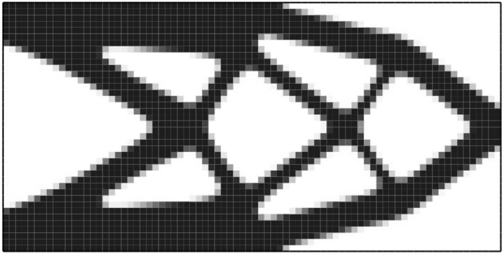
Cantilever example

Boundary conditions for an elastic cantilever: Γ_D is the left vertical side, Γ_N is the right vertical side, and Γ (dashed line) is the remaining boundary.

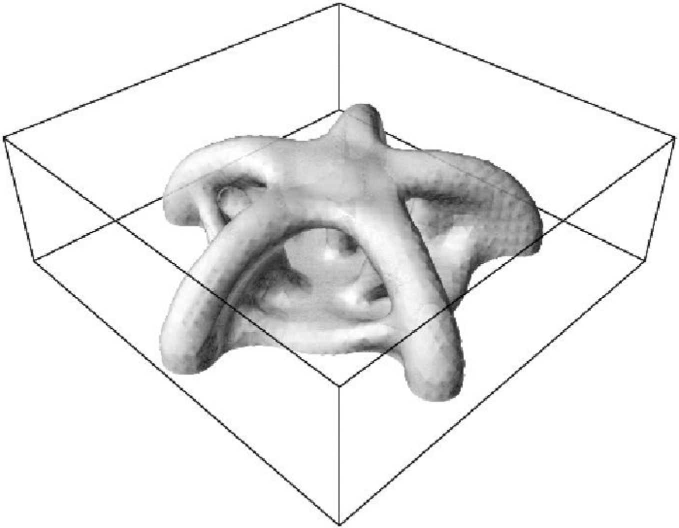


Optimal cantilever

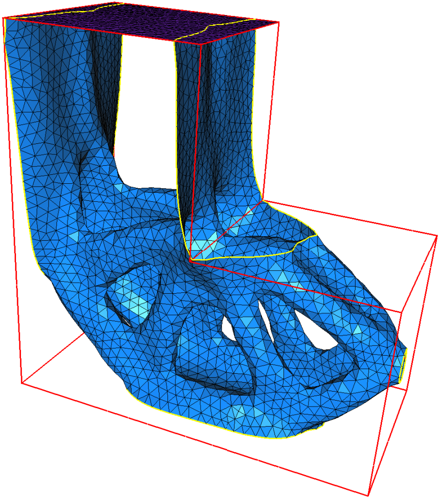




Optimal dome



L-beam

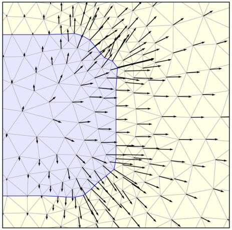


Variant of the level set method with remeshing

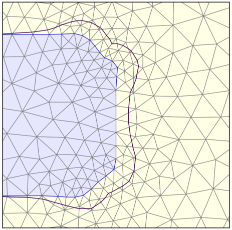
- Body-fitted mesh at each iteration thanks to the free software `Mmg3d`:
  https://www.mmgtools.org/
- After moving the level set function ψ, the mesh is adapted to fit the zero level set.
- Much more precise numerical simulations, especially for multi-physics applications.

G. Allaire, Ch. Dapogny, P. Frey, *Shape optimization with a level set based mesh evolution method*, CMAME 282, 22-53 (2014).

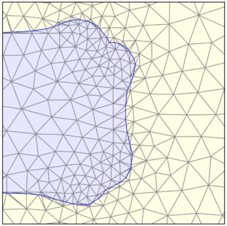
Exact remeshing with Mmg



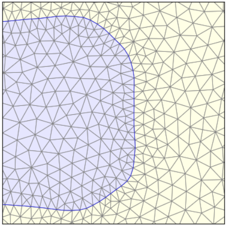
Initial interface



Zero-level set after advection



Cut mesh (bad quality)



Adapted mesh