Shape and topology optimization of structures built by additive manufacturing

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Outline of the course

- 1 - Introduction: a review of additive manufacturing
- 2 - Parametric optimization and the adjoint method
- 3 - Geometric optimization and Hadamard method
- 4 - Topology optimization and the level set method
- 5 - Typical constraints from additive manufacturing
- 6 - Optimization of lattice materials
- 7 - Coupled shape and laser path optimization

A "hot" topic with a lot of room for new ideas and modeling...

Chapter 5 - Typical constraints from additive manufacturing

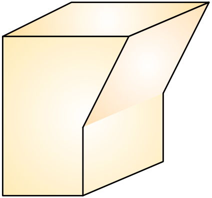
- I - Self-supported structures
- II - Support optimization
- III - Imperfect interface for supports
- IV - Residual thermal stress constraint
- V - Inherent strain model

SOFIA

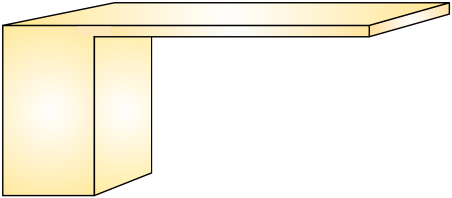
Sofia project: Add-Up, Michelin, Safran, ESI, etc. (2016-2022)

I - Self-supported structures

We study constraints for overhang limitation.



OK (left)



non-printable (right)

Overhangs lead to bad 3-d printing. They can

1. either be taken into account by adding support material,
2. or be avoided by penalization during the optimization process.

We follow the second idea.

Context of shape and topology optimization

Minimize the compliance over a set $\mathcal{U}_{ad}$ of admissibles shapes Ω

$$\min_{\Omega \in \mathcal{U}_{ad}, P(\Omega) \leq 0} J(\Omega) = \int_{\Gamma_N} g \cdot u \, ds$$

with a constraint $P(\Omega)$ to avoid overhangs.
The displacement u_Ω is the solution of

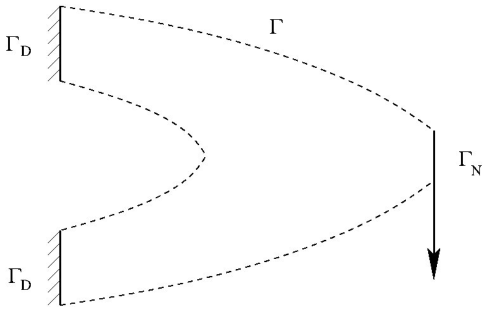
$$\begin{cases} -\operatorname{div}(A\, e(u)) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \big(A\, e(u)\big) n = g & \text{on } \Gamma_N \\ \big(A\, e(u)\big) n = 0 & \text{on } \Gamma \end{cases}$$

with the strain tensor $e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$ and the stress tensor $\sigma = Ae(u)$.

Admissible shapes

Fix $D \subset \mathbb{R}^d$ and V_0 a prescribed volume

$$\mathcal{U}_{ad} = \left\{ \Omega \subset D \text{ such that } \Gamma_D \bigcup \Gamma_N \subset \partial\Omega \text{ and } \int_\Omega dx = V_0 \right\},$$



Shape $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D$, where Γ_D and Γ_N are fixed. Only Γ is optimized (free boundary).

Geometric constraint on the normal angle

Idea: we forbid some angles of the normal to the shape with the build direction d.

For a given angle ϕ, our pointwise criterion reads

$$n(x) \cdot d \leq \cos\phi, \qquad \forall x \in \partial\Omega.$$

Denoting $(\cdot)^+ \equiv \max(\cdot, 0)$, a global penalized constraint is

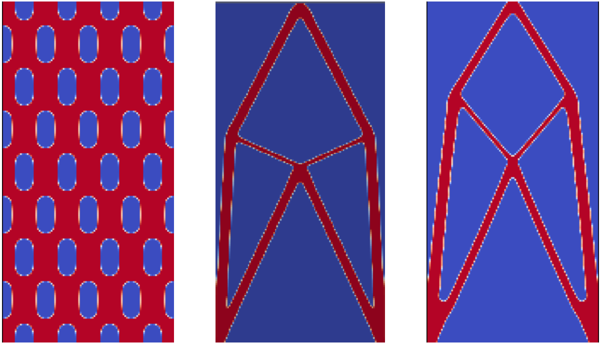
$$P(\Omega) = \int_{\partial\Omega} \left[\Big(n(s) \cdot d - \cos\phi \Big)^+ \right]^2 ds$$

Another geometric constraint

We prescribe favorable orientations of the normal, n_{g_i}, $1 \leq i \leq, m$, which correspond to the normalized gradient of m motifs.
The new constraint is

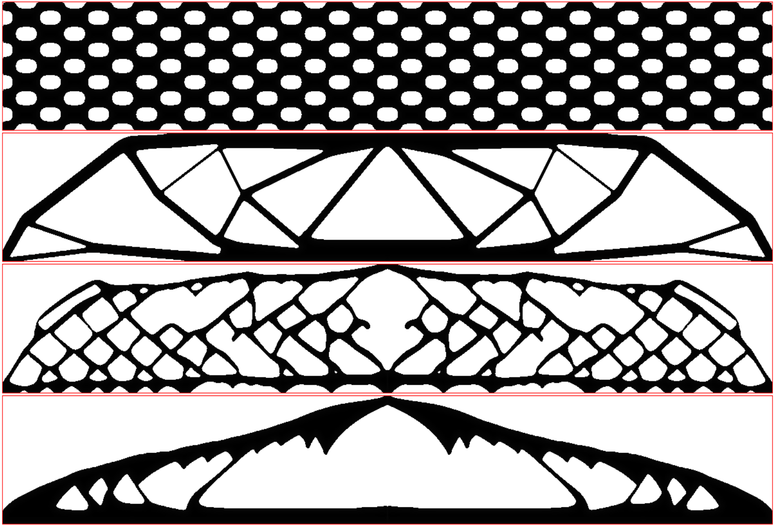
$$P(\Omega) = \int_{\partial\Omega} \prod_{i=1}^{m} |n(s) - n_{g_i}|^2 \, ds$$

Sometime it works...



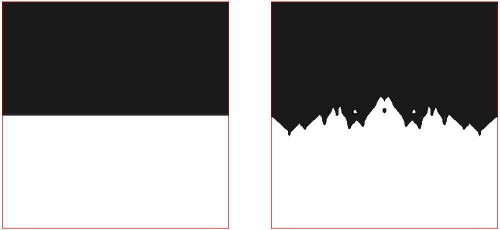
Initialization and optimal designs: without (left) and with constraint (right).
10 motif directions n_{g_i} corresponding to angles between -45° and 45°.

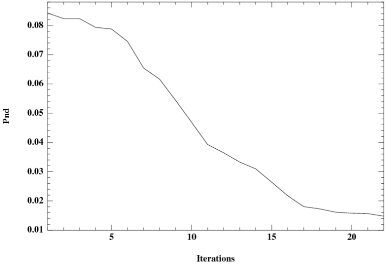
Sometime it doesn't work: dripping effect



Horizontal boundaries are replaced by oscillating boundaries.

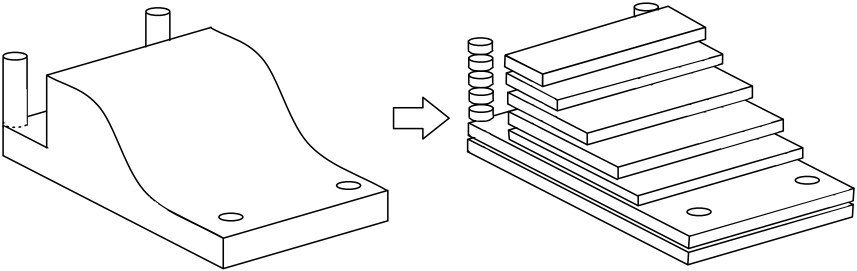
Explanation of the dripping effect





Oscillating boundaries perform better than horizontal ones for $P(\Omega)$.

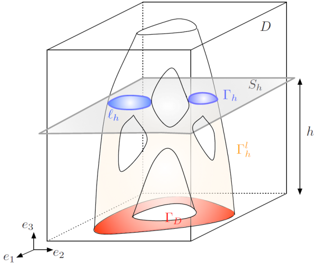
New mechanical constraint for overhang limitation



Define intermediate **"layer by layer"** shapes, for $1 \leq i \leq n$ and $h_i = Hi/n$,

$$\Omega_i = \{x \in \Omega \text{ such that } 0 < x_d < h_i\}$$

Apply self-weight to the shapes Ω_i, compute its compliance, sum them up and apply an upper bound constraint.



For each shape Ω_i, u_i solves the elasticity system:

$$\begin{cases} -\operatorname{div}(A\, e(u_i)) = \rho g & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \Gamma_D, \\ (A\, e(u_i))\, n = 0 & \text{on } \Gamma_i, \end{cases}$$

Global self-weight compliance constraint:

$$P(\Omega) = \sum_{i=1}^{n} \int_{\Omega_i} Ae(u_i) : e(u_i)\, dx = \sum_{i=1}^{n} \int_{\Omega_i} \rho g \cdot u_i\, dx$$

Self-weight compliance constraint

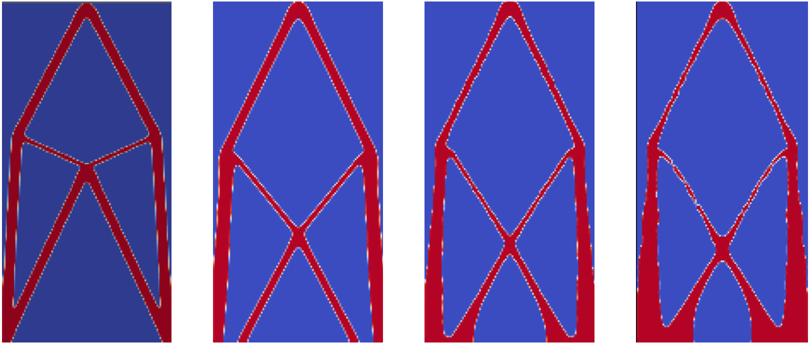
We solve the optimization problem:

$$\begin{aligned} \min_{\Omega \subset D} \quad & J(\Omega) = \int_{\Gamma_N} g \cdot u \, ds \\ \text{s.t.} \quad & V(\Omega) \leq 0.20|D| \\ & P(\Omega) \leq \alpha P(\Omega_{ref}), \ \alpha \in (0,1). \end{aligned}$$

where Ω_{ref} is the optimal design without constraint and α is a parameter of the method.

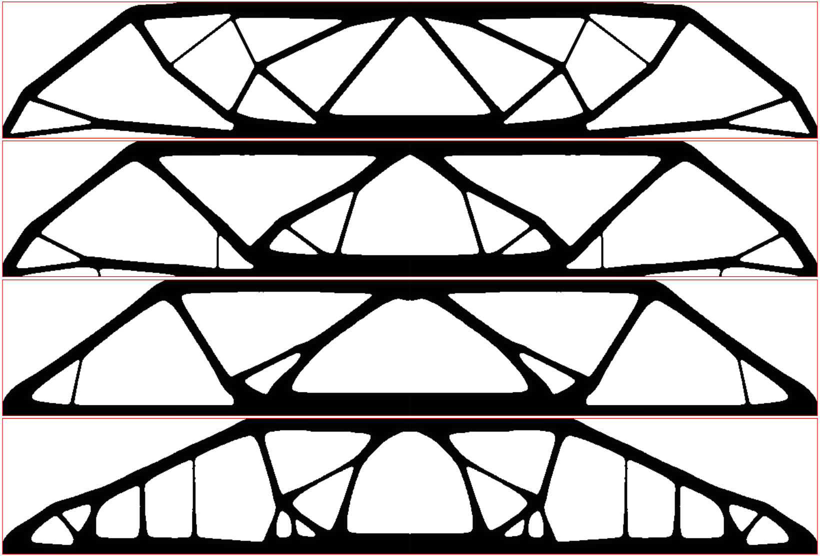
Remark: we compute the shape derivative of the constraint $P(\Omega)$ and apply a Lagrangian optimization algorithm.

Self-weight compliance constraint



Optimal designs: without constraint (left), decreasing parameter $\alpha = 0.8, 0.5, 0.3$ (right).

Self-weight compliance constraint



Upper-weight compliance constraint

Variant for a better efficiency: only the upper part of the structure is loaded.

$$g_\delta(x) = \begin{cases} g & \text{if } h_i - \delta < x_d < h_i, \\ 0 & \text{otherwise,} \end{cases}$$

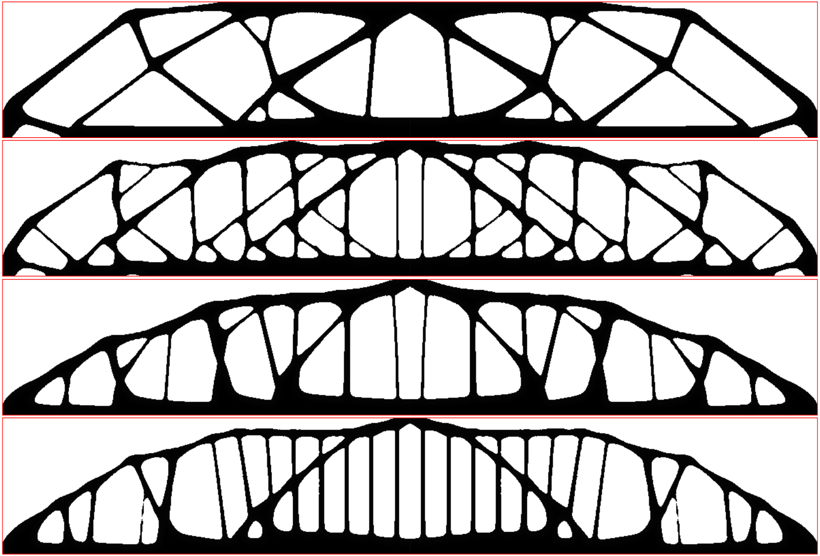
where h_i is the height of Ω_i and $\delta > 0$.
For each shape Ω_i, u_i solves the elasticity system:

$$\begin{cases} -\mathrm{div}\,(A\,e(u_i)) & = & \rho g_\delta & \text{in } \Omega_i, \\ u_i & = & 0 & \text{on } \Gamma_D, \\ (A\,e(u_i))\,n & = & 0 & \text{on } \Gamma_i, \end{cases}$$

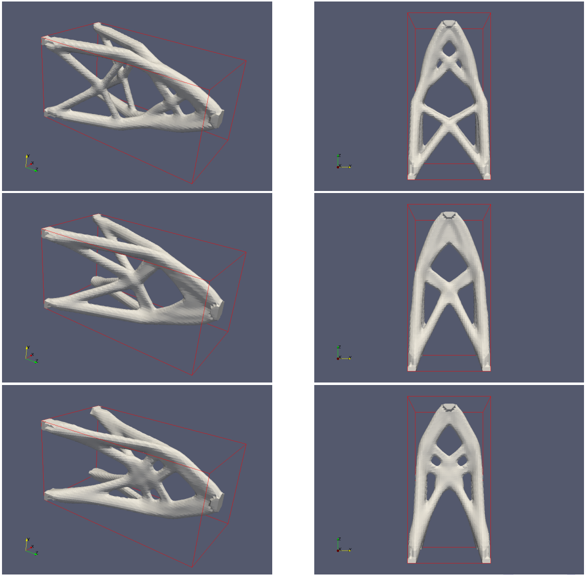
Global upper-weight compliance constraint:

$$P(\Omega) = \sum_{i=1}^{n} \int_{\Omega_i} Ae(u_i) : e(u_i)\,dx = \sum_{i=1}^{n} \int_{\Omega_i} \rho g_\delta \cdot u_i\,dx$$

Upper-weight compliance constraint



Upper-weight compliance constraint in 3-d



- Almost horizontal overhang surfaces cannot be realized by additive manufacturing.
- Nevertheless, sometime the design of the structure is imposed and cannot be changed...

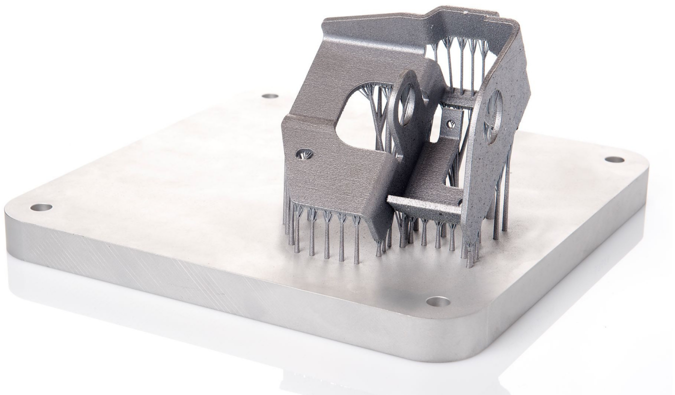
In such a case, put supports under the overhangs !

Example of supports

Supports can be full material or a lattice (perforated) material.

Example of supports

Supports can have a tree structure (Magics ®).



- they support inclined surfaces
- they fix the shape to the baseplate

Drawbacks

Impression time, additional material consumption, post-processing (removal)

The role of supports

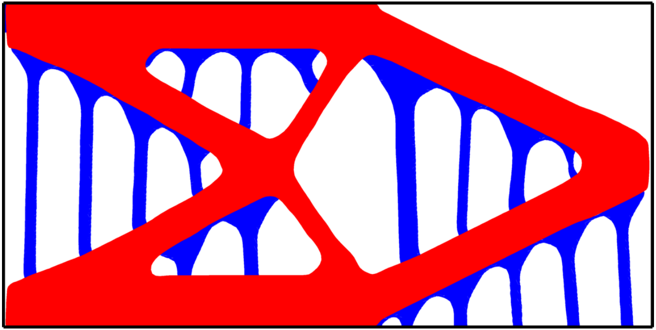
- they support inclined surfaces
- they fix the shape to the baseplate

Optimization goals

Given a certain design, to insure its successful 3D printing, optimize the orientation and topology of supports

Optimization problem

- design domain D (here, a rectangle)
- given structure $\omega \subset D$ (in red) to be printed and not optimizable
- supports $S \subset D$ (in blue) to be optimized
- mechanical model in $\Omega = \omega \cup S$
- objective function to mitigate overhangs



Typical formulation

$$\text{Minimize } J(S),$$

where $J(S)$ is related to the rigidity of the total shape $\Omega = S \cup \omega$.

- the structure ω is fixed and only the support S is optimizable
- the state equation is posed in the union $S \cup \omega$
- the material parameters may be different in ω and S
- Forces: model the "instability" of inclined regions
- Volume constraint for the support $\text{Vol}(S)$

State equation: linearized elasticity

Pseudo-gravity loads, parallel to the build direction, in ω and S:

$$\begin{cases} -\operatorname{div}\sigma = g(\rho_\omega\chi_\omega + \rho_S\chi_S) & \Omega = \omega \cup S \\ \sigma = 2\mu e(u) + \lambda \operatorname{div} u \,\mathrm{Id} & \Omega \\ e(u) = \frac{1}{2}(\nabla u + \nabla^t u) & \Omega \\ u = 0 & \Gamma_D \\ \sigma.n = 0 & \Gamma_N \end{cases}$$

Compliance minimization:

$$J(S) = \int_{\omega\cup S} g(\rho_\omega\chi_\omega + \rho_S\chi_S)\cdot u$$

where χ_ω and χ_S are the characteristic functions of ω and S. Typically $\rho_S = 0$.

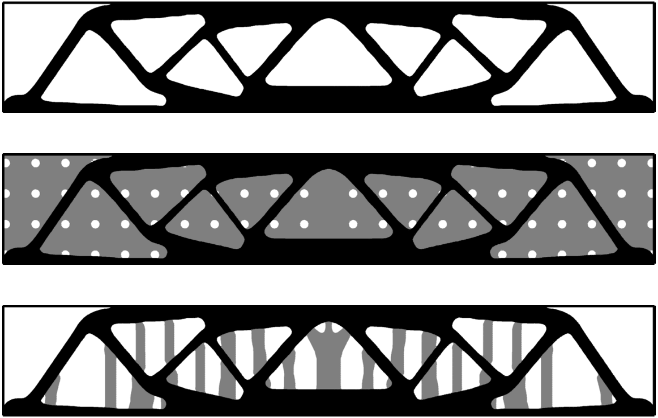
Theorem. The compliance $J(S)$ is shape differentiable and its derivative is

$$J'(S)(\theta) = \int_{\partial S \setminus \partial\omega} \left(-Ae(u) \cdot e(u) + 2\rho g \cdot u \right) \theta \cdot n \, ds$$

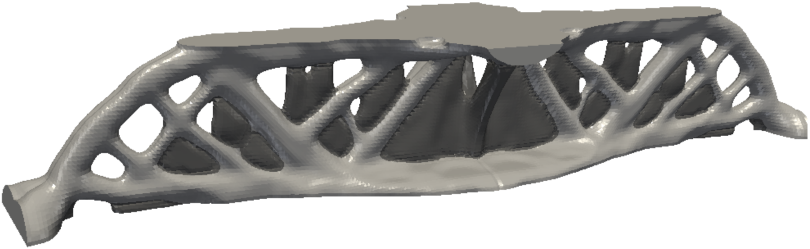
Numerical method (as in chapter 4):

- the support S is represented by a level set function
- the shape derivative is used for advecting the level set
- an augmented Lagrangian algorithm allows to take into account constraints
- FreeFem++, what else ?

MBB beam in 2D (supports in grey)

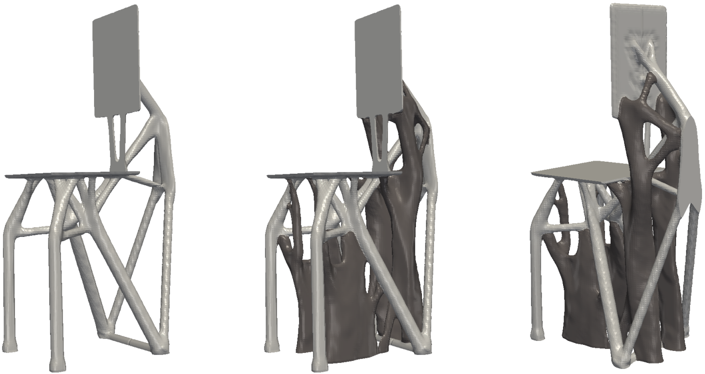


MBB beam in 3D

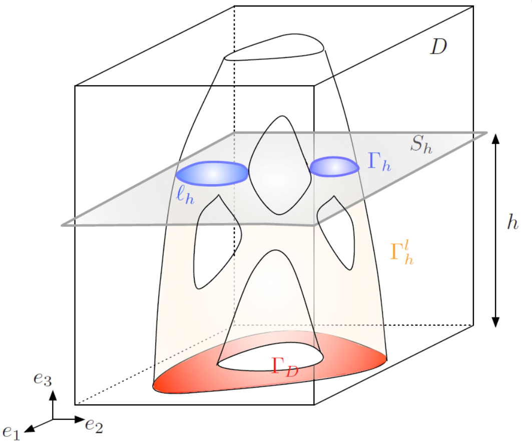


▸ Rotation

Chair in 3D



▸ Rotation ▸ Optimization

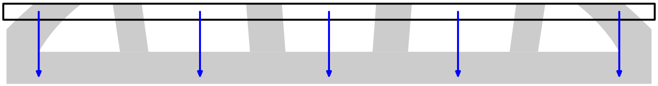


For a final shape $\Omega = \omega \cup S$, define **intermediate shapes** Ω_i of increasing height h_i

$$\Omega_i = \{x \in \Omega \text{ such that } x_d \leq h_i\} \quad 1 \leq i \leq n.$$

Multi-layer optimization

- Since the fabrication process operates layer by layer, optimize layer by layer !
- Idea already used in the previous section.
- Minimize the sum of compliances of all intermediate shapes Ω_i.
- Better modeling but higher computational cost



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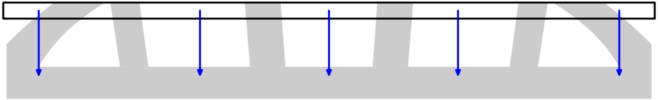

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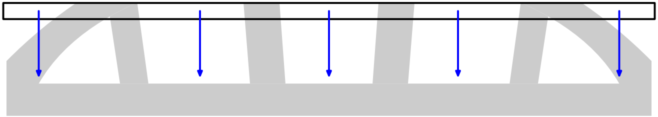

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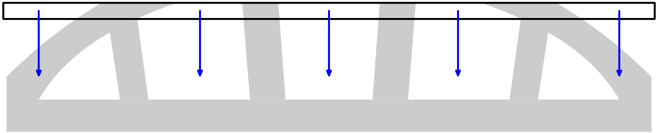

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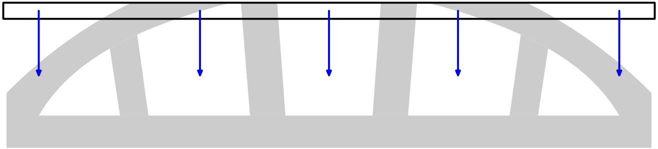

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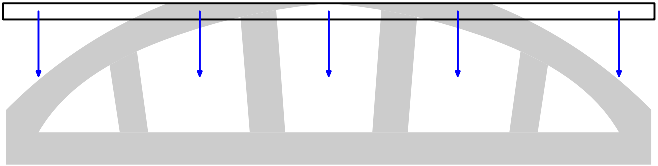

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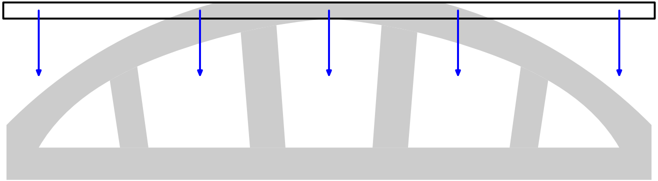

Multi-layer optimization

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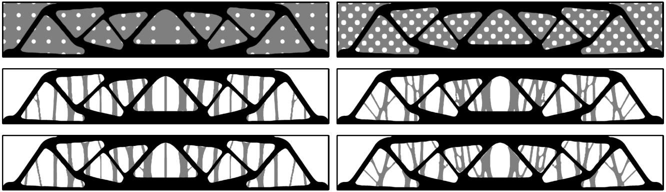
Init.

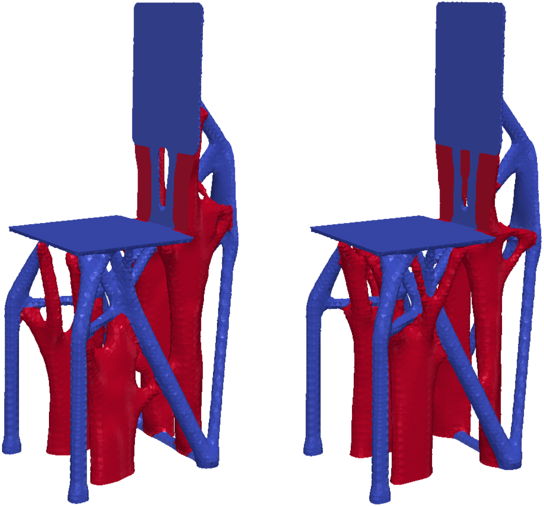
10 Slices

50 Slices



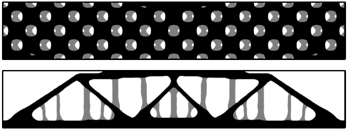
3D chair, layer by layer

5 and 10 slices

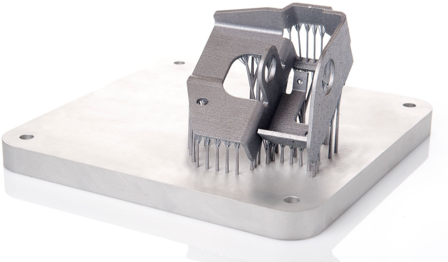


- at every iteration we solve **two** state equations : one for the final loads on the structure ω alone and another for the building loads on the supported structure $S \cup \omega$
- evolve the two shapes simultaneously using two level set functions for the parametrization
- different shape derivatives on $\partial\omega \setminus S, \partial S \setminus \omega$ and $\partial\omega \cap \partial S$

The MBB example: video

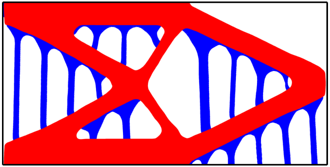


To ease the removal of supports, their contact with the built structure is made fragile on purpose.



Model of imperfect interface

- The tree structure or the dotted line of holes (for ease of separation) between the support and structure cannot be meshed exactly for macroscopic computations.
- Instead, they are modeled through an imperfect interface condition.



Setting:

- the built structure ω is fixed.
- only the support S is optimized.
- the material parameters between ω and S are the same.

Model of imperfect interface Γ

- Let Γ be the interface between ω and S with unit normal ν (outward S).
- Let u be the displacement and σ the stress tensor.
- The jump of a discontinuous quantity Q through Γ is denoted $[Q] = Q_\omega - Q_S$.

The displacement is discontinuous through Γ.

The normal stress $\sigma\nu$ is continuous through Γ.

The normal stress is a function of the displacement jump

$$R^{-1}[u] + \sigma\nu = 0$$

where R^{-1} is the rigidity of the interface.

Linear elasticity with an imperfect interface

For a smooth applied load F, the displacement u is the solution of

$$\begin{cases} -\operatorname{div}\sigma(u) = F & \text{in } \omega \text{ and in } S, \\ u = 0 & \text{on } \Gamma_D, \\ \sigma(u)n = 0 & \text{on } \Gamma_N, \\ [\sigma(u)\nu] = 0 & \text{on } \Gamma = \partial S \cap \partial\omega \\ [u] = -R\sigma(u)\cdot\nu & \text{on } \Gamma = \partial S \cap \partial\omega, \end{cases}$$

with $\sigma(u) = Ae(u) = 2\mu e(u) + \lambda \operatorname{div} u \,\mathrm{Id}$, $e(u) = \frac{1}{2}(\nabla u + \nabla^t u)$,
the jump $[f] = f_\omega - f_S$, $\nu = n_\omega$ the normal to Γ.
The matrix R is the compliance (inverse of rigidity) of the interface

$$R = \alpha(\mathrm{Id} - \nu\otimes\nu) + \beta\nu\otimes\nu,$$

where $\alpha, \beta > 0$ are the tangential and normal compliances.

Optimization problem

The **shape optimization** problem is the compliance minimization

$$\inf_{S \in \mathcal{U}_{ad}} J(S) = \int_{\omega \cup S} F \cdot u \, dx,$$

where the set of admissible supports is typically

$$\mathcal{U}_{ad} = \left\{ S \subset D \setminus \omega \text{ open set such that } \int_S dx = V_0 \right\},$$

where $D \subset \mathbb{R}^d$ is given and V_0 is a prescribed volume.

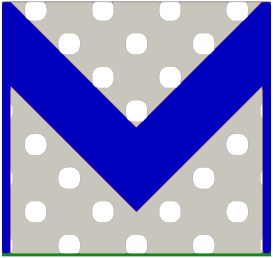
Remark. By definition the interface $\Gamma = \partial S \cap \partial \omega$ is constrained to belong to $\partial \omega$.

Theorem. Assume $\theta \cdot n = 0$ on $\partial S \cap \partial \omega$. The shape derivative of the compliance is given by

$$J'(S)(\theta) = \int_{\partial S \setminus \partial \omega} \left(-Ae(u) \cdot e(u) + 2F \cdot u \right) \; \theta \cdot n \, ds \\ - \int_{\partial(\partial S \cap \partial \omega)} R^{-1}[u] \cdot [u] \theta \cdot \tau \, dl$$

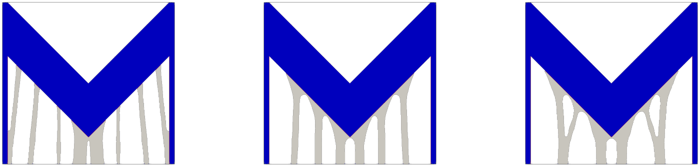
where τ is the tangent vector to $\partial \omega$, normal to $\partial S \cap \partial \omega$, and dl is the $(d-2)$ dimensional measure along $\partial(\partial S \cap \partial \omega)$.

The structure and supports are fixed on the bottom side. The force is gravity with the same material density.



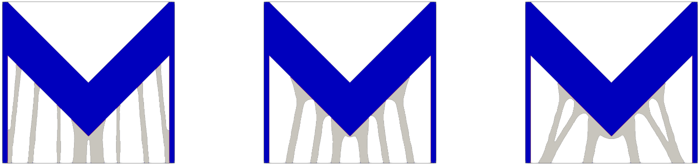
The ‘M-part’ (light blue) and its support initialization (dark blue). The domain is $D = [-1.6, 1.6]^2$ and $V(\omega) = 3.6$. The objective volume for S is $V_{sup} = 1.0$

Optimal supports for the M-structure

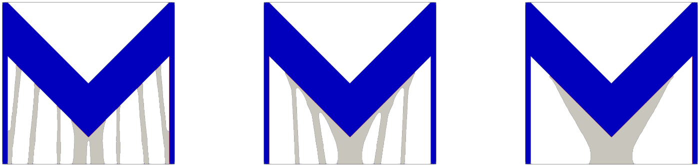


Optimized supports for $\alpha = \beta = 0.001$ (left), $\alpha = \beta = 20$ (center) and $\alpha = \beta = 50$ (right).

Tangentially weak interface



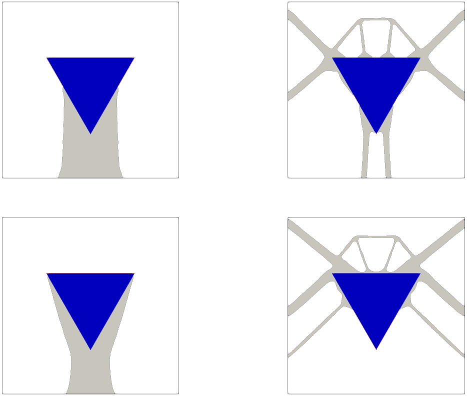
Optimized supports for $\alpha = \beta = 0.001$ (left), $\alpha = 20, \beta = 10$ (center) and $\alpha = 50, \beta = 10$ (right).



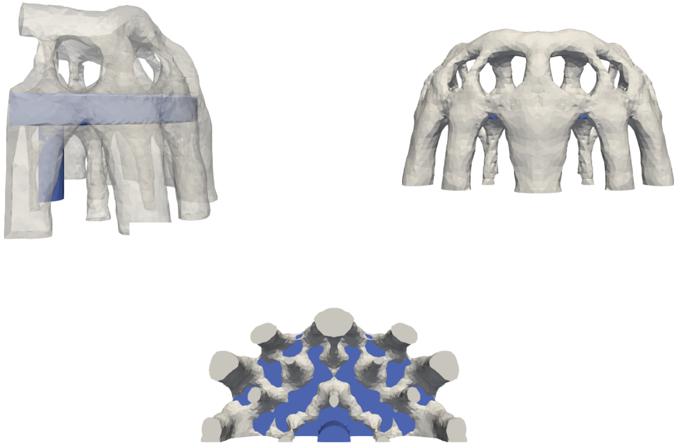
Optimized supports for $\alpha = \beta = 0.001$ (left), $\alpha = 10, \beta = 20$ (center) and $\alpha = 10, \beta = 50$ (right).

Triangular structure

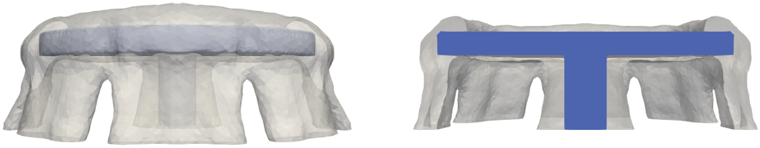
Supports can be attached to any side, except the upper one.

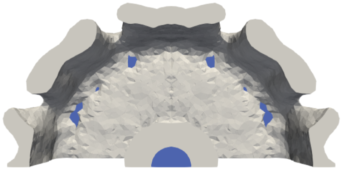


$\alpha = \beta = 0.001$ (upper left), $\alpha = \beta = 50$ (upper right),
$\alpha = 10, \beta = 50$ (lower left) and $\alpha = 50, \beta = 10$ (lower right).



weak interface $\alpha = \beta = 400$





normally weak interface $\alpha = 1$ and $\beta = 100$

IV - Residual thermal stress constraint

- Introduce intermediate **"layer by layer"** shapes $(\Omega_i)_{i=1,...,n}$.
- Each layer i is built between time t_{i-1} and t_i.
- Holes are now **filled by a metallic powder**.
- Thermal residual stress computed by a model as in
  L. Van Belle, J.-C. Boyer, G. Vansteenkiste, *Investigation of residual stresses induced during the selective laser melting process,* Key Engineering Materials, 1828-2834 (2013).
  M. Megahed, H.-W. Mindt, N. NâDri, H. Duan, O. Desmaison, *Metal additive-manufacturing process and residual stress modeling*, Integrating Materials and Manufacturing Innovation, 5:4, (2016).

1st state equation for the final shape

For a given applied load $f : \Gamma_N \to \mathbb{R}^d$,

$$\begin{cases} -\operatorname{div}(A\,e(u_{final})) = 0 & \text{in } \Omega \\ u_{final} = 0 & \text{on } \Gamma_D \\ \big(A\,e(u_{final})\big)n = f & \text{on } \Gamma_N \\ \big(A\,e(u_{final})\big)n = 0 & \text{on } \Gamma \end{cases}$$

Objective function: compliance

$$J(\Omega) = \int_{\Gamma_N} f \cdot u_{final}\,dx,$$

2nd state equation for the intermediate shapes

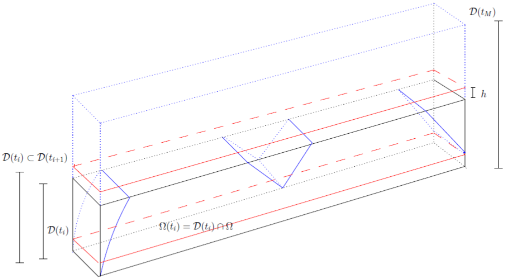
Heat equation:

$$\begin{cases} \rho \dfrac{\partial T}{\partial t} - \mathrm{div}(\lambda \nabla T) = Q(t) & \text{in } (t_{i-1}, t_i) \times D_i \\ T = T_{init} & \text{on } (t_{i-1}, t_i) \times \Gamma_{base} \\ \lambda \nabla T \cdot n = -H_e(T - T_{init}) & \text{on } (t_{i-1}, t_i) \times (\partial D_i \setminus \Gamma_{base}) \\ T(t = t_{i-1}) = T_{init} & \text{in } D_i \setminus D_{i-1} \end{cases}$$

Thermoelastic quasi-static equation:

$$\begin{cases} -\mathrm{div}(\sigma) = 0 & \text{and } \sigma = \sigma^{el} + \sigma^{th} \quad \text{in } (t_{i-1}, t_i) \times D_i, \\ \sigma^{el} = Ae(u) & \text{and } \sigma^{th} = K(T - T_{init})\,\mathrm{Id}, \end{cases}$$

Notations



1. Each layer i is built between time t_{i-1} and t_i, $1 \leq i \leq n$.
2. Build chamber D, vertical build direction e_d.
3. Intermediate domains $D_i = \{x \in D \text{ such that } x_d \leq h_i\}$.
4. Final shape Ω and intermediate shapes $\Omega_i = \Omega \cap D_i$.
5. Mixture $D_i = \Omega_i \cup P_i$ of solid and powder.

The objective function is

$$J(\Omega) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_{D_i} j(u, \sigma, T)\, dx\, dt$$

where (u, σ, T) is the displacement, stress and temperature fields for the **intermediate shapes**. A constraint on the compliance of the final shape is imposed

$$C(\Omega) = \int_{\Omega} f \cdot u_{final}\, dx \leq C(\Omega_{ref}),$$

where u_{final} is the elastic displacement for the **final shape**

$$-\operatorname{div}(A\, e(u_{final})) = f \quad \text{in } \Omega$$

The shape derivative of $J(\Omega)$ is computed by an adjoint method.

Example for an objective $j(u)$ (without T and σ for simplicity).
Elasticity adjoint equation: no "backward effect"

$$-\operatorname{div}\left(e(\eta)\right) = -j'(u) \quad \text{in } (t_{i-1}, t_i) \times D_i$$

Adjoint heat equation: backward in time, from $i = n$ to 1,

$$\begin{cases} \rho \dfrac{\partial p}{\partial t} + \operatorname{div}(\lambda \nabla p) = K \operatorname{div} \eta & \text{in } (t_{i-1}, t_i) \times D_i \\ p = 0 & \text{on } (t_{i-1}, t_i) \times \Gamma_{base} \\ \lambda \nabla p \cdot n = -H_e p & \text{on } (t_{i-1}, t_i) \times (\partial D_i \setminus \Gamma_{base}) \\ p(t = t_n) = 0 & \text{in } D_n \end{cases}$$

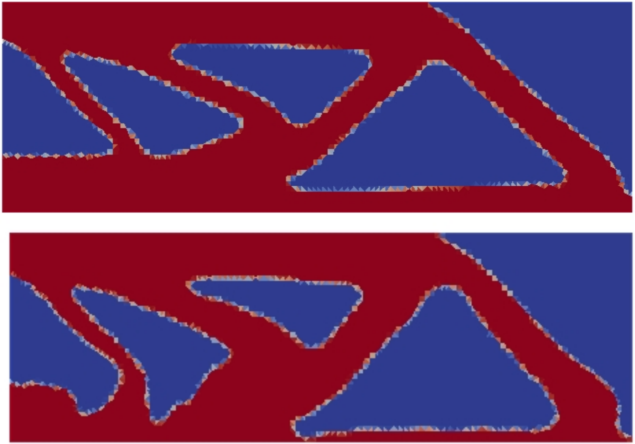
Reversed order of coupling: **first**, solve the adjoint elasticity, **second**, the adjoint heat equation.

- Half MBB beam (2-d).
- **Full model with 20 layers and 5 time steps per layer.**
- Minimize the deviatoric part of the stress $\sigma_D = 2\mu e(u)_D$

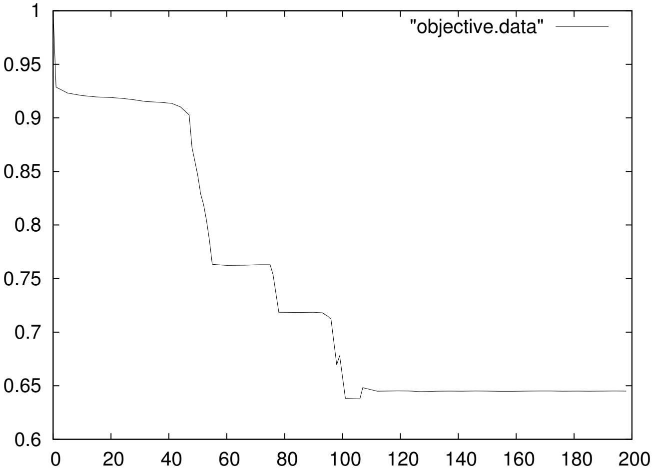
$$J_1(\Omega) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_D |\sigma_D|^2 \, dx \, dt$$

- Constraints on volume (fixed) and compliance.
- Initial design: optimal design for compliance minimization.

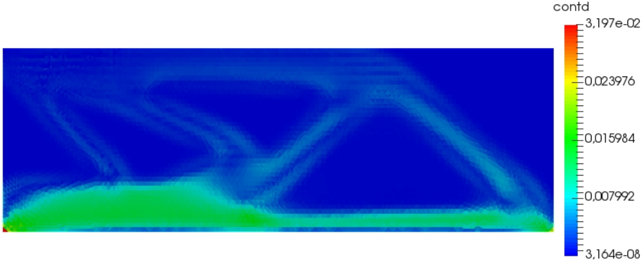
Initial (top) and final (bottom) shape



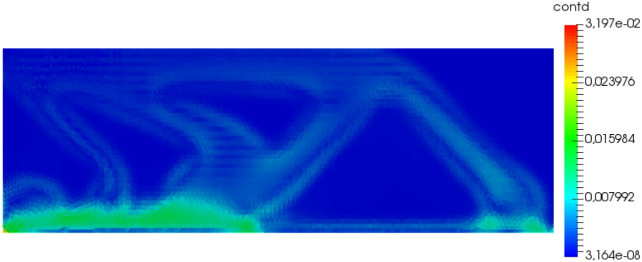
Convergence history (thermal stress)



Plot of thermal stress $\sqrt{\int_0^T |\sigma^D|^2(x)\, dt}$



initial



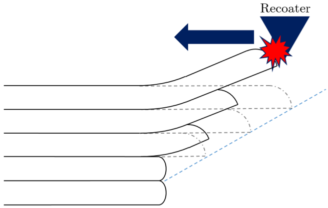
final

V - Inherent strain model

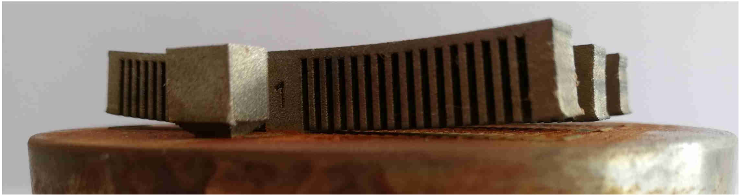
We now optimize supports to minimize thermal deformations.

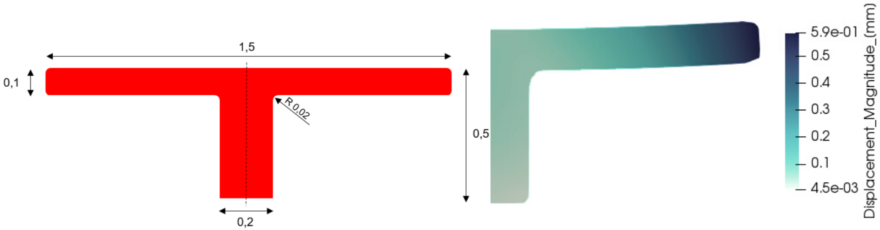
It requires a thermo-mechanical model. For example:

- thermo-elasticity and heat equation (previous section),
- inherent strain model.



Sketch of the layer deformation, which can stop the layer deposition, because of thermal retraction upon cooling.





Geometry of T-shape (left), vertical displacement (right) induced by the fabrication process (simulation of a thermo-elastic model).

A well-known model for welding process. No heat equation !

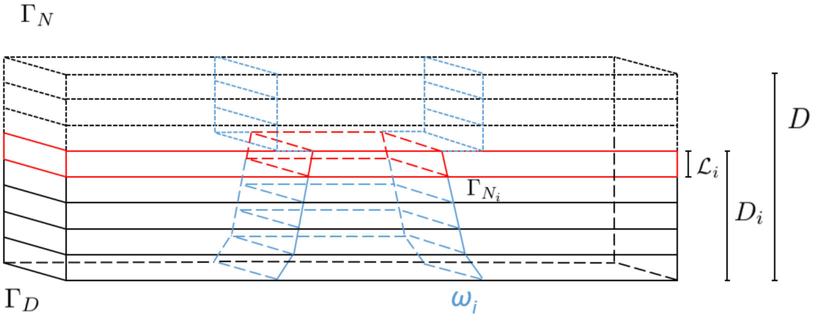
The thermal effects are encoded in a given inherent strain tensor ϵ^*.

Solve the standard quasi-static elasticity equations with a stress tensor defined by

$$\sigma = \sigma^{el} + \sigma^{inh} \quad \text{with} \quad \sigma^{el} = Ae(u) \text{ and } \sigma^{inh} = A\epsilon^*.$$

The inherent strain tensor is calibrated by an inverse problem on a test case. Typically

$$\epsilon^* = \begin{bmatrix} -0.0001 & 0 & 0 \\ 0 & -0.0001 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Layer by layer construction of the part ω in the build chamber D.

M. Bihr, G. Allaire, X. Betbeder-Lauque, B. Bogosel, F. Bordeu, J. Querois, *Part and supports optimization in metal powder bed additive manufacturing using simplified process simulation*, CMAME 395, 114975 (2022).

The supported structure $\Omega = \omega \cup S$ is divided into M layers, and each intermediate shape is built from the first i layers such that $\Omega_i = \Omega \cap D_i$. The model is

$$\begin{cases} -\mathrm{div}(\sigma_i) &= 0 & \text{in } \Omega_i, \\ \sigma_i &= A\left(e(u_i) + \epsilon^*_{\mathcal{L}_i}\right) & \text{with } \epsilon^*_{\mathcal{L}_i}(x) = \epsilon^* \chi_{\mathcal{L}_i}(x), \\ \sigma_i n &= 0 & \text{on } \Gamma_{N_i}, \\ u_i &= 0 & \text{on } \Gamma_D \cap \partial\Omega_i. \end{cases}$$

We consider a criterion

$$J(S) = \sum_{i=1}^{M} \int_{\Omega_i} j(u_i)dx \quad \text{with} \quad j(u_i) = |\max(0, u_i \cdot e_d)|^2 \chi_{\mathcal{L}_i}.$$

The optimization problem is

$$\begin{array}{ll} \min\limits_{S \subset D \setminus \omega} & J(S) \\ \text{such that} & |S| = |S_0|, \end{array}$$

Introduce an adjoint state p_i solution of

$$\begin{cases} -\mathrm{div}(Ae(p_i)) & = & -j'(u_i) & \text{in } \Omega_i, \\ (Ae(p_i))n & = & 0 & \text{on } \Gamma_{N_i}, \\ p_i & = & 0 & \text{on } \Gamma_D. \end{cases}$$

Proposition. The shape derivative in the direction of the vector field $\theta \in W^{1,\infty}(D, \mathbb{R}^d)$ is given by

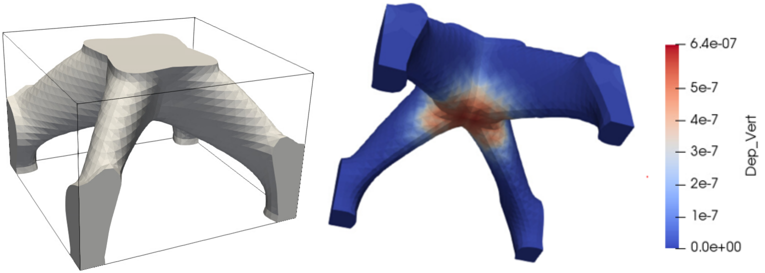
$$J'(S)(\theta) = \sum_{i=1}^{M} \int_{\partial S \cap D_i} \theta \cdot n \Big(j(u_i) + A \left(e(u_i) + \epsilon^*_{\mathcal{L}_i} \right) : e(p_i) \Big) \, ds.$$

Proof. Introducing the Lagrangian

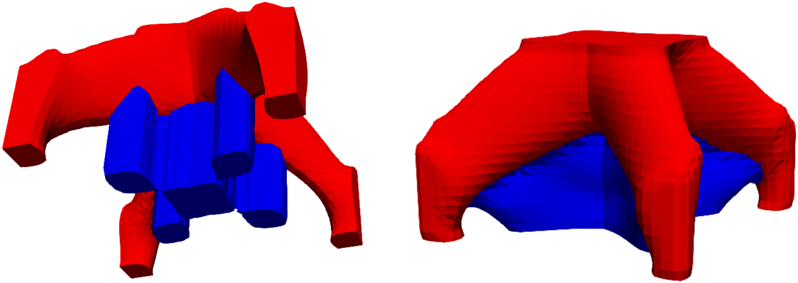
$$\mathcal{L}(\Omega, \{u_i\}, \{p_i\}) = \sum_{i=1}^{M} \int_{\Omega_i} j(u_i) \, dx + \sum_{i=1}^{M} \int_{\Omega_i} A \left(e(u_i) + \epsilon^*_{\mathcal{L}_i} \right) : e(p_i) \, dx$$

and differentiating $\mathcal{L}$ with respect to all the variables give the desired result.

Structure to be built

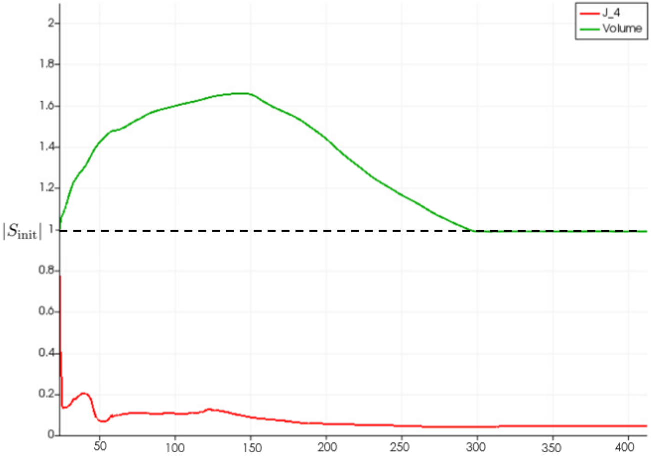


Fixed part ω to build (left) and associated vertical displacements predicted by the inherent strain model (right).



Supports S in blue: initial ones (left) and optimized ones (right) for the fixed part ω in red.

Convergence history



Convergence history for the objective function $J(S)$ (red) and the volume $|S|$ (green).

Comparison of deformations for an optimized and a non-optimized structure. Calibration of the inherent strain model.